S-Parameters for Three and Four Two-Port Networks

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Abstract: *The scattering parameters are fundamental in the characterization of electrical devices at high frequencies. They are particularly useful for analyzing and designing multiport high frequency and microwave networks. This paper presents explicit mathematical formulas for the resultant S-parameters for parallel, series, and cascade of three and four two-port networks in terms of the S-parameters of the individual two-port networks. The mathematical formulas are derived using the signal flow graph (SFG) method.*

I. Introduction

The scattering parameters formalism is one of the most useful subjects that microwave engineers need to fully comprehend due to its characterization of microwave networks. Elementary circuit analysis provides many methods for describing *n*-port networks. These methods include impedance parameters, admittance parameters, transmission line, and hybrid parameters [1]-[3]. These methods best describe dc and low-frequency circuits. Particularly, they are difficult to apply to microwave networks. To characterize such high-frequency networks, one can employ scattering parameters (or S-parameters) in place of impedance (Z) or admittance (Y). They are used in defining the performance of many electrical devices, such as microwave devices, filters, transformers, and amplifiers. The S-parameters are defined in terms of wave variables, which are more easily measured at high frequencies than are voltage and current. However, the S-parameters are not directly suitable for the analysis of parallel and series network of two or more two-port networks. Such parallel, series, and cascade networks are usually analyzed by multiplying the individual matrices using the Y-parameters, Z-parameters, T-parameters, respectively. It should be noted that there are different definitions of other parameters in the literature [4]-[6]. Since many microwave networks consist of parallel, series, and cascade connections of both passive and active elements, it is desirable to have explicit formulas directly involving the S-parameters. Such formulas are useful for analyzing waveguide and microstrip discontinuities. In this paper, we provide the formulas for three and four of parallel, series, and cascade

two-port networks. The problem of determining the S-parameters for parallel, series, and cascade two-ports can be solved in using the signal flow graph (SFG). Using the Y and Z matrices will involve knowing in advance the reference characteristic of admittance Y_0 and impedance Z_0, respectively. Therefore, we choose to use the SFG approach.

The S-parameters are taught in courses in microwave engineering at the undergraduate and graduate levels. However, textbooks on microwave engineering do not cover the analysis of three or four two-port networks. Therefore, this paper will supplement such microwave engineering textbook and help in covering three and four two-port networks.

This paper presents explicit formulas for the resultant S-parameter for three and four two-port networks connected of parallel, series, and cascade in terms of the S-parameters of the individual two-ports. The formulas are derived by SFG and verified using simulation. In section 2, we present the background information on S-parameters for two-port networks. Section 3 deals with the derivation for the S-parameter analysis, while section 4 is on conclusion.

II. Background

The scattering (or S-) parameters are fixed properties of the linear circuit. They describe how the energy couples between each pair of ports (or transmission lines) connected to the circuit. For a two-port network (with one input and one output), the matrix equations are

$$b_1 = S_{11}a_1 + S_{12}a_2 \quad (1)$$

$$b_2 = S_{21}a_1 + S_{22}a_2 \quad (2)$$

where a_1 and a_2 are the incident waves, while b_1 and b_2 are the scattered (reflected) waves, as illustrated in Fig. 1. In matrix form, equations (1) and (2) can be written as

$$[b] = [S][a] \quad (3)$$

where

$$[S] = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}, [b] = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, [a] = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \quad (4)$$

Figure 2 shows the flow graph for a two-port network. From equations (1) and (2), the S-parameter are obtained as

$$S_{11} = \left.\frac{b_1}{a_1}\right|_{a_2=0}$$ **= Input reflection coefficient at port 1** **(5)**

$$S_{21} = \left.\frac{b_2}{a_1}\right|_{a_2=0}$$ **= Direct transfer ratio at port 1 to port 2** **(6)**

$$S_{22} = \left.\frac{b_2}{a_2}\right|_{a_1=0}$$ **= Output reflection coefficient at port 2** **(7)**

and

$$S_{12} = \left.\frac{b_1}{a_2}\right|_{a_1=0}$$ **= Reverse transfer ratio at port 2 to port 1** **(8)**

III. Derivations

The most common and suitable method used for the analysis of parallel two-port networks is Y-parameters (admittance parameters) such that

$$[a] = [Y][b] \tag{9}$$

Or

$$a_1 = Y_{11}b_1 + Y_{12}b_2 \tag{10}$$

$$a_2 = Y_{21}b_1 + Y_{22}b_2 \tag{11}$$

The Y-parameters of overall networks are the sum of the individual Y parameters of the individual networks. This means that

$$[Y] = [Y_a] + [Y_b] + ... + [Y_n] \tag{12}$$

where *n* is the number of two-port networks. An example for *n* =3 is shown in Fig. 3.

As mentioned earlier, the SFG is a method of writing a set of equations, whereby the related variables are represented by points and the interrelations by directed lines giving a direct picture of a signal flow. The main advantage of such a graphical technique in solving parallel or series networks are the convenient pictorial representation and the painless method of proceeding directly to the solution from the graph. With that preliminary background, we now consider the parallel two-port networks as in Fig. 4, that shows the S-parameter SFG for three parallel two-port networks. We find the following S-parameters:

For the parallel connection of three two-port networks:

$$S_{11} = S_{11}^{a} + S_{11}^{b} + S_{11}^{c} \quad (13)$$

$$S_{12} = S_{12}^{a} + S_{12}^{b} + S_{12}^{c} \quad (14)$$

$$S_{21} = S_{21}^{a} + S_{21}^{b} + S_{21}^{c} \quad (15)$$

$$S_{22} = S_{22}^{a} + S_{22}^{b} + S_{22}^{c} \quad (16)$$

For the parallel connection of four two-port networks:

$$S_{11} = S_{11}^{a} + S_{11}^{b} + S_{11}^{c} + S_{11}^{d} \quad (17)$$

$$S_{12} = S_{12}^{a} + S_{12}^{b} + S_{12}^{c} + S_{12}^{d} \quad (18)$$

$$S_{21} = S_{21}^{a} + S_{21}^{b} + S_{21}^{c} + S_{21}^{d} \quad (19)$$

$$S_{22} = S_{22}^{a} + S_{22}^{b} + S_{22}^{c} + S_{22}^{d} \quad (20)$$

we can observe that the general rule for the parallel connection of *n* two-port networks:

$$S_{ij} = \sum_{k=1}^{n} S_{ij}^{k} \quad (21)$$

k **is a, b,...,k two-port networks**

$i, j = 1$ **or** 2.

The most common and suitable method used for the analysis of series two-port networks is Z-parameters (impedance parameters) such that

$$[b] = [Z][a] \quad (22)$$

or

$$b_1 = Z_{11}a_1 + Z_{12}a_2 \quad (23)$$

$$b_2 = Z_{21}a_1 + Z_{22}a_2 \quad (24)$$

The Z-parameters of overall networks are the sum of the individual Z parameters of the individual networks. This means that

$$[Z]=[Z_a]+[Z_b]+.....+[Z_n] \quad (25)$$

where *n* is the number of two-ports networks such as in Fig. 5.

We now consider the series of two-port networks as in Fig. 6, that shows the S-parameter SFG for three series-connected two-port networks. We find the following S-parameters:

For the series connection of three two-port networks:

$$S_{11}=\frac{S_{11}^{a}S_{11}^{b}S_{11}^{c}}{1-S_{12}^{a}S_{21}^{b}-S_{12}^{b}S_{21}^{c}+S_{12}^{a}S_{12}^{b}S_{21}^{b}S_{21}^{c}} \quad (26)$$

$$S_{12}=S_{12}^{c} \quad (27)$$

$$S_{21}=S_{21}^{a} \quad (28)$$

$$S_{22}=\frac{S_{22}^{a}S_{22}^{b}S_{22}^{c}}{1-S_{12}^{a}S_{21}^{b}-S_{12}^{b}S_{21}^{c}+S_{12}^{a}S_{12}^{b}S_{21}^{b}S_{21}^{c}} \quad (29)$$

For the series connection of four two-port networks:

$$S_{11}=\frac{S_{11}^{a}S_{11}^{b}S_{11}^{c}S_{11}^{d}}{1-S_{12}^{a}S_{21}^{b}-S_{12}^{b}S_{21}^{c}-S_{12}^{c}S_{21}^{d}+S_{12}^{a}S_{12}^{b}S_{21}^{b}S_{21}^{c}+S_{12}^{a}S_{12}^{c}S_{21}^{b}S_{21}^{d}+S_{12}^{b}S_{12}^{c}S_{21}^{c}S_{21}^{d}-S_{12}^{a}S_{12}^{b}S_{12}^{c}S_{21}^{b}S_{21}^{c}S_{21}^{d}} \quad (30)$$

$$S_{12}=S_{12}^{d} \quad (31)$$

$$S_{21}=S_{21}^{a} \quad (32)$$

$$S_{22}=\frac{S_{22}^{a}S_{22}^{b}S_{22}^{c}S_{22}^{d}}{1-S_{12}^{a}S_{21}^{b}-S_{12}^{b}S_{21}^{c}-S_{12}^{c}S_{21}^{d}+S_{12}^{b}S_{12}^{c}S_{21}^{c}S_{21}^{d}+S_{12}^{a}S_{12}^{b}S_{21}^{b}S_{21}^{c}+S_{12}^{a}S_{12}^{c}S_{21}^{b}S_{21}^{d}-S_{12}^{a}S_{12}^{b}S_{12}^{c}S_{21}^{b}S_{21}^{c}S_{21}^{d}} \quad (33)$$

We now relate the S-parameters to the chain transfer parameters (also known as the transfer scattering parameters) or simply T-parameters, which are suitable for the analysis of cascaded two-ports. It should be noted that there are different definitions of the T-parameter in the literature [4-5]. Here we follow the type described by Hewlett Parkard [5] since it is the most common.

$$\begin{pmatrix} b_1 \\ a_1 \end{pmatrix}=\begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}\begin{pmatrix} a_2 \\ b_2 \end{pmatrix} \quad (34)$$

By manipulating the matrices in equations (1), (2), and (34) we can readily show that

$$T = \begin{pmatrix} -\frac{\Delta S}{S_{21}} & \frac{S_{11}}{S_{21}} \\ -\frac{S_{22}}{S_{21}} & \frac{1}{S_{21}} \end{pmatrix}, \qquad S = \begin{pmatrix} \frac{T_{12}}{T_{22}} & \frac{\Delta T}{T_{22}} \\ \frac{1}{T_{22}} & -\frac{T_{21}}{T_{22}} \end{pmatrix} \qquad \textbf{(35)}$$

where $\Delta S = S_{11}S_{22} - S_{12}S_{21}$, the determinant of the S-matrix, and $\Delta T = T_{11}T_{22} - T_{12}T_{21}$, the determinant of the T-matrix.

With that preliminary background, we now consider three cascaded two-ports as shown in Fig. 7. The chain matrix of the overall cascade connection can be written as

$$\begin{pmatrix} b_1 \\ a_1 \end{pmatrix} = T \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} \qquad \textbf{(36)}$$

where

$$T = T_a T_b T_c \qquad \textbf{(36a)}$$

or

$$\begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} = \begin{pmatrix} T_{11}^a & T_{12}^a \\ T_{21}^a & T_{22}^a \end{pmatrix} \begin{pmatrix} T_{11}^b & T_{12}^b \\ T_{21}^b & T_{22}^b \end{pmatrix} \begin{pmatrix} T_{11}^c & T_{12}^c \\ T_{21}^c & T_{22}^c \end{pmatrix} \qquad \textbf{(36b)}$$

For example,

$$T_{22} = T_{21}^a T_{11}^b T_{12}^c + T_{21}^a T_{12}^b T_{22}^c + T_{22}^a T_{21}^b T_{12}^c + T_{22}^a T_{22}^b T_{22}^c \qquad \textbf{(37)}$$

Substituting for the individual T-matrices using eq. (34) yields

$$\begin{aligned} S_{21} &= \frac{1}{T_{22}} = \frac{1}{T_{21}^a T_{11}^b T_{12}^c + T_{21}^a T_{12}^b T_{22}^c + T_{22}^a T_{21}^b T_{12}^c + T_{22}^a T_{22}^b T_{22}^c} \\ &= \frac{1}{-\frac{S_{22}^a}{S_{21}^a}\left(-\frac{\Delta S^b}{S_{21}^b}\right)\frac{S_{11}^c}{S_{21}^c} - \frac{S_{22}^a}{S_{21}^a}\frac{S_{11}^b}{S_{21}^b}\frac{1}{S_{21}^c} - \frac{1}{S_{21}^a}\frac{S_{22}^b}{S_{21}^b}\frac{S_{11}^c}{S_{21}^c} + \frac{1}{S_{21}^a}\frac{1}{S_{21}^b}\frac{1}{S_{21}^c}} \\ &= \frac{S_{21}^a S_{21}^b S_{21}^c}{\left[1 - S_{22}^a S_{11}^b - S_{22}^b S_{11}^c + S_{22}^a \Delta S^b S_{11}^c\right]} \end{aligned} \qquad \textbf{(38)}$$

where $\Delta S^b = S_{11}^b S_{22}^b - S_{12}^b S_{21}^b$, the determinant of the S^b-matrix. By taking a similar approach for other elements of the overall S-matrix, we obtain

$$S_{11} = S_{11}^a + \frac{S_{12}^a S_{21}^a (S_{11}^b - \Delta S^b S_{11}^c)}{\left[1 - S_{22}^a S_{11}^b - S_{22}^b S_{11}^c + S_{22}^a \Delta S^b S_{11}^c\right]} \qquad \textbf{(39)}$$

$$S_{12} = \frac{S_{12}^a S_{12}^b S_{12}^c}{\left[1 - S_{22}^a S_{11}^b - S_{22}^b S_{11}^c + S_{22}^a \Delta S^b S_{11}^c\right]} \quad \textbf{(40)}$$

$$S_{22} = S_{22}^c + \frac{(S_{22}^b - S_{22}^a \Delta S^b) S_{12}^c S_{21}^c}{\left[1 - S_{22}^a S_{11}^b - S_{22}^b S_{11}^c + S_{22}^a \Delta S^b S_{11}^c\right]} \quad \textbf{(41)}$$

As mentioned earlier, the signal flow graph (SFG) can be used to derive the same formulas. SFG is a method of writing a set of equations, whereby the related variables are represented by points and the interrelations by directed lines giving a direct picture of signal flow. The main advantage of such a graphical technique in solving cascaded networks is the convenient pictorial representation and the method of proceeding directly to the solution from the graph. The cascaded connection of three two-ports in Fig. 7. is represented by the flow graph of Fig. 8. With Fig. 8, we can derive eq. (38)-(41).

Similar steps can be taken to obtain the S-parameters for four cascaded two-ports. The results are

$$S_{11} = S_{11}^a + \frac{S_{12}^a S_{21}^a \left(S_{11}^b - \Delta S^b S_{11}^c + \Delta S^b \Delta S^c S_{11}^d - S_{11}^b S_{22}^c S_{11}^d\right)}{\left[1 - S_{22}^a S_{11}^b - S_{22}^b S_{11}^c - S_{22}^c S_{11}^d + S_{22}^a \Delta S^b S_{11}^c - S_{22}^a \Delta S^b \Delta S^c S_{11}^d + S_{22}^b \Delta S^c S_{11}^d + S_{22}^a S_{11}^b S_{22}^c S_{11}^d\right]} \quad \textbf{(42)}$$

$$S_{12} = \frac{S_{12}^a S_{12}^b S_{12}^c S_{12}^d}{\left[1 - S_{22}^a S_{11}^b - S_{22}^b S_{11}^c - S_{22}^c S_{11}^d + S_{22}^a \Delta S^b S_{11}^c - S_{22}^a \Delta S^b \Delta S^c S_{11}^d + S_{22}^b \Delta S^c S_{11}^d + S_{22}^a S_{11}^b S_{22}^c S_{11}^d\right]} \quad \textbf{(43)}$$

$$S_{21} = \frac{S_{21}^a S_{21}^b S_{21}^c S_{21}^d}{\left[1 - S_{22}^a S_{11}^b - S_{22}^b S_{11}^c - S_{22}^c S_{11}^d + S_{22}^a \Delta S^b S_{11}^c - S_{22}^a \Delta S^b \Delta S^c S_{11}^d + S_{22}^b \Delta S^c S_{11}^d + S_{22}^a S_{11}^b S_{22}^c S_{11}^d\right]} \quad \textbf{(44)}$$

$$S_{22} = S_{22}^d + \frac{S_{12}^d S_{21}^d \left(S_{22}^c - S_{22}^b \Delta S^c + S_{22}^a \Delta S^b \Delta S^c - S_{22}^a S_{11}^b S_{22}^c\right)}{\left[1 - S_{22}^a S_{11}^b - S_{22}^b S_{11}^c - S_{22}^c S_{11}^d + S_{22}^a \Delta S^b S_{11}^c - S_{22}^a \Delta S^b \Delta S^c S_{11}^d + S_{22}^b \Delta S^c S_{11}^d + S_{22}^a S_{11}^b S_{22}^c S_{11}^d\right]} \quad \textbf{(45)}$$

where $\Delta S^c = S_{11}^c S_{22}^c - S_{12}^c S_{21}^c$**, the determinant of the** S^c**-matrix**

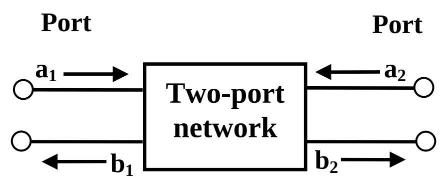

Figure 1. Incoming and outgoing waves for a two-port network.

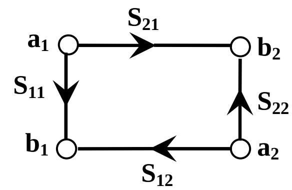

Figure 2. SFG for a two-port network.

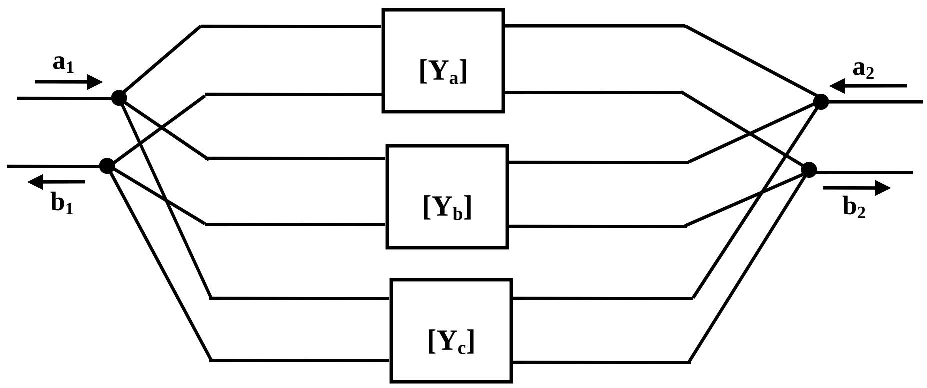

Figure 3. Parallel connection of three two-port networks.

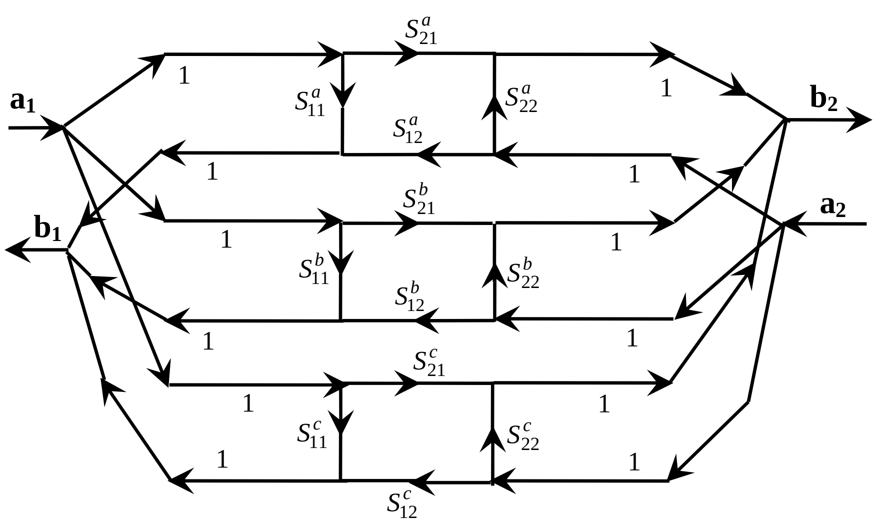

Figure 4. SFG for three parallel two-port networks.

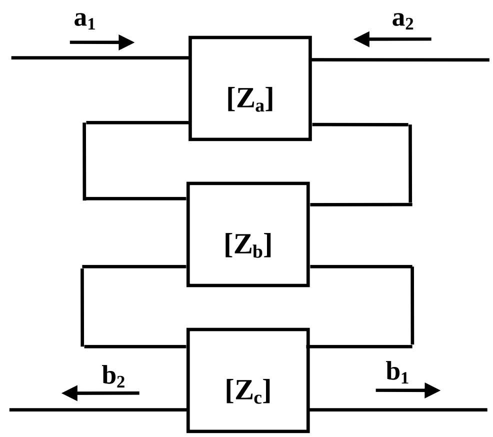

Figure 5. Series connection of three two-port networks.

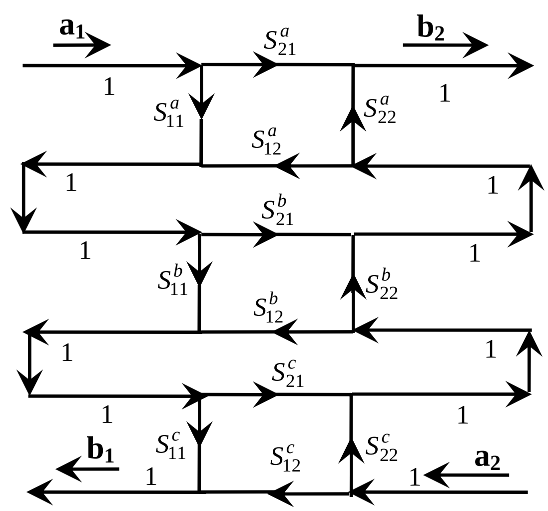

Figure 6. SFG for three series connection of two-port networks.

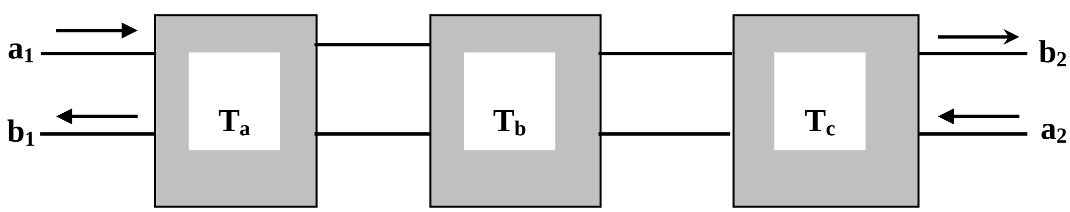

Figure 7. Cascaded connection of three two-port networks.

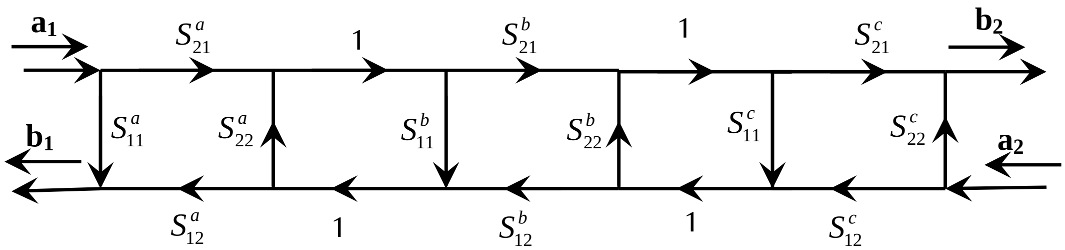

Figure 8. SFG for three cascaded two-port networks.

IV. Conclusion

Explicit mathematical formulas for finding the composite S-parameters for three and four two-port networks connected in parallel, series, and cascade have been derived in this paper. The mathematical formulas are derived using SFG.

V. References

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