Non-Calculus Derivation of the Maximum Power Transfer Theorem

by

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Abstract: In Electrical Engineering Technology textbooks, the maximum power transfer theorem is usually stated and then maybe demonstrated with experimental or simulated evidence. The derivation is not given, because it is thought that calculus is needed. In this paper, we present a derivation of this theorem without using calculus. The method requires the student to know the algebraic rules of inequalities. The theorem for DC circuits is presented for two cases: (i) finding the optimum value of the load resistor for maximum power transfer to it, and (ii) finding the optimum value of the internal resistance of the source, for maximum power transfer to the load. We also give non-calculus proofs for AC circuits for three cases: (i) when we are free to choose the complete load impedance, i.e. both the load resistor and the load reactance, (ii) when we are free only to choose the load resistor because the load reactance is fixed, and (iii) when we are free to choose the magnitude of the load impedance but the angle of the load impedance is fixed.

I. Introduction

In Electrical Engineering Technology textbooks, the maximum power transfer theorem is usually stated and then maybe demonstrated with experimental or simulated evidence. The derivation is not given, because it is thought that calculus is needed. In this paper, we present a derivation of this theorem without using calculus. The method is different from that of Paul and Gardner [1], which is also derived without calculus. (For completeness, we review their method in the Appendix). The advantage to our method is that it can be generalized to the constrained load case for AC circuits, as we will discover in Section III. Our method requires the student to know the algebraic rules of inequalities.

The theorem for DC circuits will be derived first in Section II, where we present two cases: (i) finding the optimum value of the load resistor for maximum power transfer to it, and (ii) finding the optimum value of the internal resistance of the source, for maximum power transfer to the load. In Section III, we give proofs for AC circuits for three cases: (i) when we are free to choose the complete load impedance, i.e. both the load resistor and the load reactance, (ii) when we are free only to choose the load resistor because the load reactance is fixed, and (iii) when we are free to choose the magnitude of the load impedance but the angle of the load impedance is fixed. Case (ii) is discussed in Appendix G of [2, pp. 1138-1139]. Case (iii) is discussed in [3, pg. 475].

II. Derivation of the Maximum Power Transfer Theorem for DC Circuits

Consider Fig. 1 which shows a Thevenin DC source which is connected to a load resistor R_L. The question which the maximum power transfer theorem answers is twofold: (i) what should the value of R_L be if we want the load resistor to absorb maximum power from the source, and (ii) what should the value of the internal resistance be if we want maximum power transfer to the load. The latter question is usually not dealt with in many textbooks. Nonetheless, the question is important and is answered by PSpice simulation in Exercise 4.4 of [4, pp. 213-214]. Also, it is given as Problem 4.86 in [3, pg.175].

For both cases, we need to find an expression for the power to the load, which can be found as given below.

Using the voltage divider rule, the voltage across the load in Fig.1 is given by

$$V_L = \frac{R_L}{R_L + R} E . \tag{1}$$

Hence, the power absorbed by the load resistor is

$$\begin{aligned} P_L &= \frac{V_L^2}{R_L} \\ &= \frac{1}{R_L}\left(\frac{R_L}{R_L + R} E\right)^2 \\ &= \frac{R_L}{(R_L + R)^2} E^2. \end{aligned} \tag{2}$$

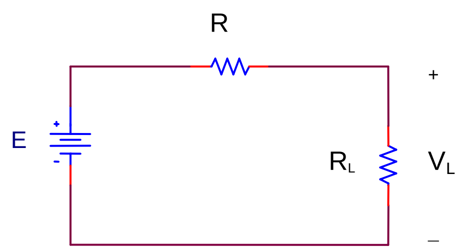

Fig. 1. Practical DC voltage source with load resistor, R_L. R is the internal resistance of the source.

A. Value of the Load Resistor which Gives Maximum Power Transfer to the Load

In this subsection, we will derive the maximum power transfer theorem for the case when the internal resistor of the DC source is fixed and we are free to choose the load resistor R_L . We will do this in two ways. The first way (Method I) is the more straightforward way, but it is limited to this purely resistive case. The second way (Method II) follows a procedure that can be generalized to allow the derivation of the theorem for AC circuits, as detailed in Section III B and Section III C. Also, Method II uses a key result that is developed in Method I.

Method I

Let $R_L = kR$: so, from (2),

$$P_L = \frac{kR}{\left(kR + R\right)^2} E^2. \tag{3}$$

However, by factoring out R in the denominator, (3) can be rewritten as

$$P_L = \frac{k}{\left(1+k\right)^2} \frac{E^2}{R}. \tag{4}$$

Clearly then, to find the maximum value of (4), we must find the maximum value of

$$f = \frac{k}{\left(1+k\right)^2} = \frac{1}{k + k^{-1} + 2}. \tag{5}$$

Keep in mind that k is positive. Clearly then, to maximize (5), we must minimize $k + k^{-1}$. Fortunately, it is straightforward to find the minimum value of $k + k^{-1}$ without calculus. In fact, $k + k^{-1} \geq 2$. To prove this we use the inequality $\left(k-1\right)^2 \geq 0$. Hence, $k^2 - 2k + 1 \geq 0$ or $k - 2 + k^{-1} \geq 0$. Rearranging the latter inequality gives the desired result, i.e. $k + k^{-1} \geq 2$.

Therefore, $k = 1$ minimizes $k + k^{-1}$ which then maximizes (4). Hence, $R_L = R$, as required. Additionally, substituting $k = 1$ into (5) gives the maximum power transferred as

$$P_{L\max} = \frac{1}{\left(1+1\right)^2} \frac{E^2}{R} = \frac{E^2}{4R}. \tag{6}$$

Summary 1: For a totally resistive circuit, the load resistor should equal the internal resistance of the source, for maximum power transfer to the load. Furthermore, the actual maximum power transferred to the load will then be given by (6).

To illustrate this, we use (2) to plot the absorbed power as a function of the load resistance. This plot is given in Fig. 2, where we have assumed that the internal resistance of the source is $R = 1\,\Omega$ and the applied voltage is E = 1 V. Clearly, the maximum power absorbed is 0.25 W in agreement with (6) and occurs when $R_L = 1\,\Omega$, as it should.

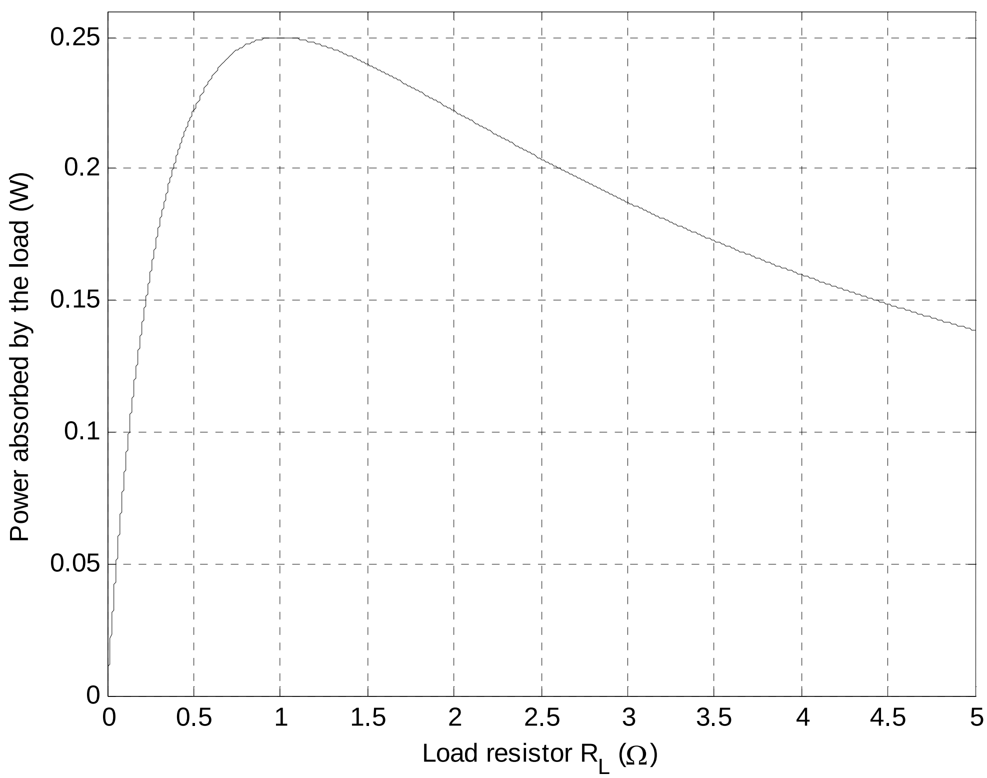

Fig. 2. Power absorbed as a function of the load resistance, assuming that the internal resistance is $1\,\Omega$ and the applied voltage is E = 1 V.

Method II

As mentioned above, we want this method to be able to be generalized to AC circuits with constrained loads. Therefore, we will be presenting the general approach simultaneously with the purely resistive case.

Step 1.

In this second method, we begin by expanding the denominator of (2) to get,

$$\begin{aligned} P_L &= \frac{R_L}{R_L^2 + 2RR_L + R^2} E^2 \\ &= \frac{x}{x^2 + bx + c} E^2, \end{aligned} \tag{7}$$

where $b = 2R$ and $c = R^2$ in this case.

This means that the denominator of the power equation is a quadratic in the variable we wish to optimize ($x = R_L$ in this case).

Step 2.

Next, we divide the numerator and the denominator of this power equation by the constant coefficient c of the quadratic in the denominator of the power equation (R^2 in this case). Hence, (7) becomes

$$\begin{aligned} P_L &= \frac{\frac{R_L}{R^2}}{\frac{R_L^2}{R^2} + \frac{2RR_L}{R^2} + 1} E^2, \\ &= \frac{\frac{x}{c}}{\frac{x^2}{c} + \frac{bx}{c} + 1} E^2. \end{aligned} \tag{8}$$

Step 3.

We now define $k^2 = \frac{R_L^2}{R^2}$ or $k = \frac{R_L}{R}$. This means that (8) can be written as

$$\begin{aligned} P_L &= \frac{\frac{R_L}{R^2}}{\frac{R_L^2}{R^2} + \frac{2RR_L}{R^2} + 1} E^2 \\ &= \frac{k/R}{k^2 + \frac{2R}{R}k + 1} E^2 \\ &= \frac{k}{k^2 + 2k + 1} \frac{E^2}{R} \\ &= \frac{1}{k + k^{-1} + 2} \frac{E^2}{R}. \end{aligned} \tag{9}$$

In the general case, we define $k^2 = \frac{x^2}{c}$ or $k = \frac{x}{\sqrt{c}}$. Therefore, (8) becomes

$$\begin{aligned} P_L &= \frac{k/\sqrt{c}}{k^2 + \frac{b}{\sqrt{c}}k + 1} E^2 \\ &= \frac{k}{k^2 + \frac{b}{\sqrt{c}}k + 1} \frac{E^2}{\sqrt{c}} \\ &= \frac{1}{k + k^{-1} + \frac{b}{\sqrt{c}}} \frac{E^2}{\sqrt{c}}. \end{aligned} \tag{10}$$

As argued earlier in Method I, $k = 1$ minimizes (9) and (10).

We will show in Sections III B and C that when this general procedure is followed, we are able to find the optimum value of the load variable that maximizes the power transferred to the load for AC circuits.

B. Value of the Internal Resistor which Gives Maximum Power Transfer to the Load

In this subsection, we answer the following question: suppose the load resistor is already specified, but we are free to choose the value of the internal resistor of the source; what then should this value be for maximum power transfer to the load?

Again, we begin with (2) which can be written as

$$P_L = \frac{1}{\left(1+\frac{R}{R_L}\right)^2}\frac{E^2}{R_L}. \tag{11}$$

Clearly, to maximize (11), we need to minimize $\left(1+\frac{R}{R_L}\right)^2$, which will be achieved with $R = 0$. Hence, for maximum power transfer to the load, the internal resistance of the source should be zero, i.e. we want an ideal voltage source.

Setting $R = 0$ gives us the maximum power that can be absorbed by the load, i.e.

$$P_{L\max} = \frac{E^2}{R_L}. \tag{12}$$

Summary 2: For a totally resistive circuit, the internal resistance of the source should be chosen to be zero, for maximum power transfer to the load. Furthermore, the actual maximum power transferred to the load will then be given by (12).

To illustrate this, we use (11) to plot the absorbed power as a function of the internal resistance. This plot is given in Fig. 3, where we have assumed that the load resistance is $R = 1\,\Omega$ and the applied voltage is E = 1 V. Clearly, the maximum power absorbed is 1 W in agreement with (12) and occurs when $R_L = 0\,\Omega$, as it should.

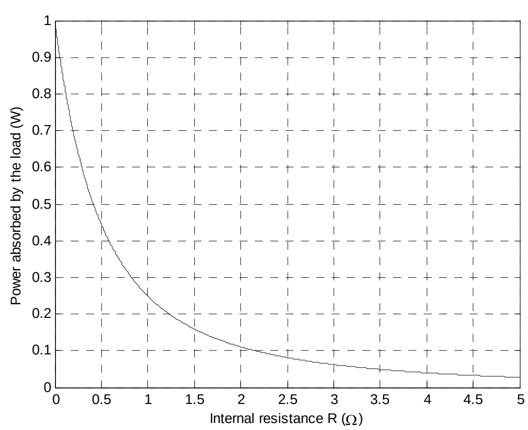

Fig. 3. Power absorbed as a function of the internal resistance, assuming that the load resistance is $1\,\Omega$ and the applied voltage is E = 1 V.

III. Derivation of the Maximum Power Transfer Theorem for AC Circuits

In this section, we will derive the maximum power transfer theorem for AC circuits, without the use of calculus for three cases:

(i) when we are free to choose the complete load impedance, i.e. both the load resistor and the load reactance,

(ii) when we have no control of the reactive portion of the load (as discussed in Appendix G of [2, pp. 1138-1139]), and

(iii) when we have no control over the phase of the load impedance, but we are free to choose the magnitude of the load (as discussed in [3, pg. 475]).

For all three cases, we will be considering Fig. 4 which shows a Thevenin AC source which is connected to a load impedance Z_L .

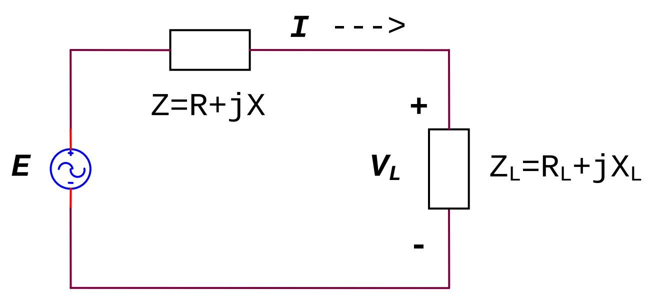

Fig. 4. Practical AC voltage source with load impedance, $Z_L = R_L + jX_L$. The internal impedance of the source is $Z = R + jX$.

Using Ohm's Law, the current phasor, $\boldsymbol{I}$, through the load in Fig.4 is given by

$$\boldsymbol{I} = \frac{E\angle 0^o}{Z + Z_L} = \frac{E\angle 0^o}{R + R_L + j\left(X + X_L\right)} . \tag{13}$$

Hence, the power absorbed by the load resistor is

$$\begin{aligned} P_L &= \left|\boldsymbol{I}\right|^2 R_L \\ &= \frac{R_L}{\left(R + R_L\right)^2 + \left(X + X_L\right)^2} E^2 . \end{aligned} \tag{14}$$

A. Maximum Power Transfer Theorem with No Restriction on the Load Impedance

In the first case, we want to know what should the value of Z_L be if we want the load impedance to absorb maximum power from the source. In this case, we are free to choose both the load resistance and load reactance.

Clearly, to maximize (14), $(X + X_L)^2$ should be minimized. This was also recognized in [5, pg. 718] and [6, pg. 382]. Hence, we make the load reactance the negative of the internal (Thevenin) reactance, i.e. $X_L = -X$. With this choice, (14) becomes (2), which was maximized with $R_L = R$. Therefore, it is clear that the load impedance should be equal to the conjugate of the Thevenin impedance of the source, as is well-known from the calculus derivation.

Furthermore, the maximum power transferred is also given by (6), where E is now the root-mean-square (*rms*) value of the source.

Summary 3: The load impedance should be chosen to be equal to the conjugate of the internal impedance of the source, for maximum power transfer to the load. Furthermore, the actual maximum power transferred to the load will then be given by (6), where E is now the *rms* magnitude of the voltage source *E*.

To illustrate this, we use (14) to plot the absorbed power as a function of the load resistance and the load reactance. This plot is given as a contour graph in Fig. 5, where we have assumed that the internal impedance of the source is $Z = 1 + j1\,\Omega$ and the applied voltage is $\boldsymbol{E} = 1\angle 0\,\text{V}$. Clearly, the maximum power absorbed is 0.25 W in agreement with (6) and occurs when $Z_L = 1 - j1\,\Omega$, as it should.

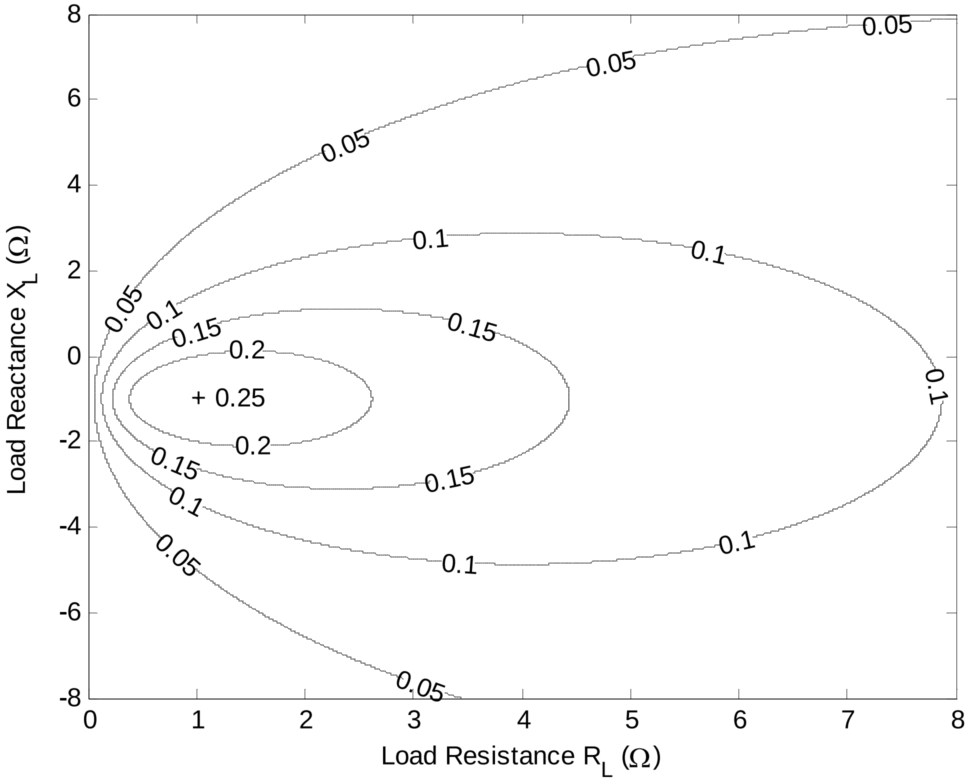

Fig. 5. Contour graph of the power absorbed by the load as the load reactance and load resistance are varied. The internal impedance of the source is $Z = 1 + j1\,\Omega$ and $\boldsymbol{E} = 1\angle 0$ V. The numbers on the contour lines are power levels. Clearly, the maximum absorbed power is 0.25 W and occurs when $Z_L = 1 - j1\,\Omega$.

B. Maximum Power Transfer Theorem when Load Reactance cannot be Varied, but the Load Resistance can be Chosen

In this second case, we are free to choose the optimum value of R_L, but the value of X_L cannot be changed. To find the optimum value of R_L, we write (14) as

$$P_L = \frac{R_L}{\left(R+R_L\right)^2 + \left(X + X_L\right)^2} E^2$$

$$= \frac{R_L}{R_L^2 + 2RR_L + R^2 + \left(X + X_L\right)^2} E^2. \qquad (15)$$

Dividing the numerator and denominator of (15) by $R^2 + \left(X + X_L\right)^2$ gives

$$P_L = \frac{\dfrac{R_L}{R^2+\left(X+X_L\right)^2}}{\dfrac{R_L^2+2RR_L}{R^2+\left(X+X_L\right)^2}+1}E^2. \tag{16}$$

However, we can rewrite (16) as

$$P_L = \frac{\dfrac{R_L}{\sqrt{R^2+\left(X+X_L\right)^2}}}{\dfrac{R_L^2}{R^2+\left(X+X_L\right)^2}+2\dfrac{R}{\sqrt{R^2+\left(X+X_L\right)^2}}\dfrac{R_L}{\sqrt{R^2+\left(X+X_L\right)^2}}+1}\frac{E^2}{\sqrt{R^2+\left(X+X_L\right)^2}}. \tag{17}$$

Defining $k = \dfrac{R_L}{\sqrt{R^2+\left(X+X_L\right)^2}}$, (17) becomes

$$\begin{aligned} P_L &= \frac{k}{k^2+2\dfrac{R}{\sqrt{R^2+\left(X+X_L\right)^2}}k+1}\frac{E^2}{\sqrt{R^2+\left(X+X_L\right)^2}} \\ &= \frac{1}{k+k^{-1}+2\dfrac{R}{\sqrt{R^2+\left(X+X_L\right)^2}}}\frac{E^2}{\sqrt{R^2+\left(X+X_L\right)^2}}. \end{aligned} \tag{18}$$

Clearly, in order for (18) to be a maximum, $k+k^{-1}$ must be a minimum. However, we have already established that $k=1$ accomplishes this. Hence, for maximum power transfer to the load, $k = \dfrac{R_L}{\sqrt{R^2+\left(X+X_L\right)^2}} = 1$, or $R_L = \sqrt{R^2+\left(X+X_L\right)^2}$, which of course, is the same result obtained with calculus.

Additionally, we can obtain the maximum power that is transferred to the load from (18) and the fact that $k=1$. Hence,

$$P_{L\max} = \frac{1}{2+2\frac{R}{\sqrt{R^2+\left(X+X_L\right)^2}}}\frac{E^2}{\sqrt{R^2+\left(X+X_L\right)^2}} = \frac{E^2}{2\left(\sqrt{R^2+\left(X+X_L\right)^2}+R\right)}. \quad (19)$$

This result is also given in Appendix G of [2, pp. 1138-1139], which was obtained by calculus. Note that if we are free to make $\left(X+X_L\right)^2=0,$ or $X_L=-X$, (19) becomes (6) as it should.

Summary 4: If the load reactance cannot be varied, the load resistance should be chosen to be equal to the magnitude of the impedance formed by the internal resistance and the total reactance of the circuit, i.e. $R_L = \sqrt{R^2+\left(X+X_L\right)^2}$, for maximum power transfer to the load. Furthermore, the actual maximum power transferred to the load will then be given by (19), where E is now the *rms* magnitude of the voltage source *E*.

To illustrate this, we use (14) to plot the absorbed power as a function of the load resistance, with the load reactance being a fixed value. This plot is given in Fig. 6, where we have assumed that the internal impedance of the source is $Z=1+j1\Omega$ and the applied voltage is $\boldsymbol{E}=1\angle 0$ V. Clearly, the optimum value of the load resistor is well-predicted by $R_L = \sqrt{R^2+\left(X+X_L\right)^2}$ **,** whereas the maximum power absorbed is well-predicted by (19). Furthermore, when the load reactance is its optimum, i.e. $X_L=-1\Omega$, the power absorbed is its maximum value, as it should be.

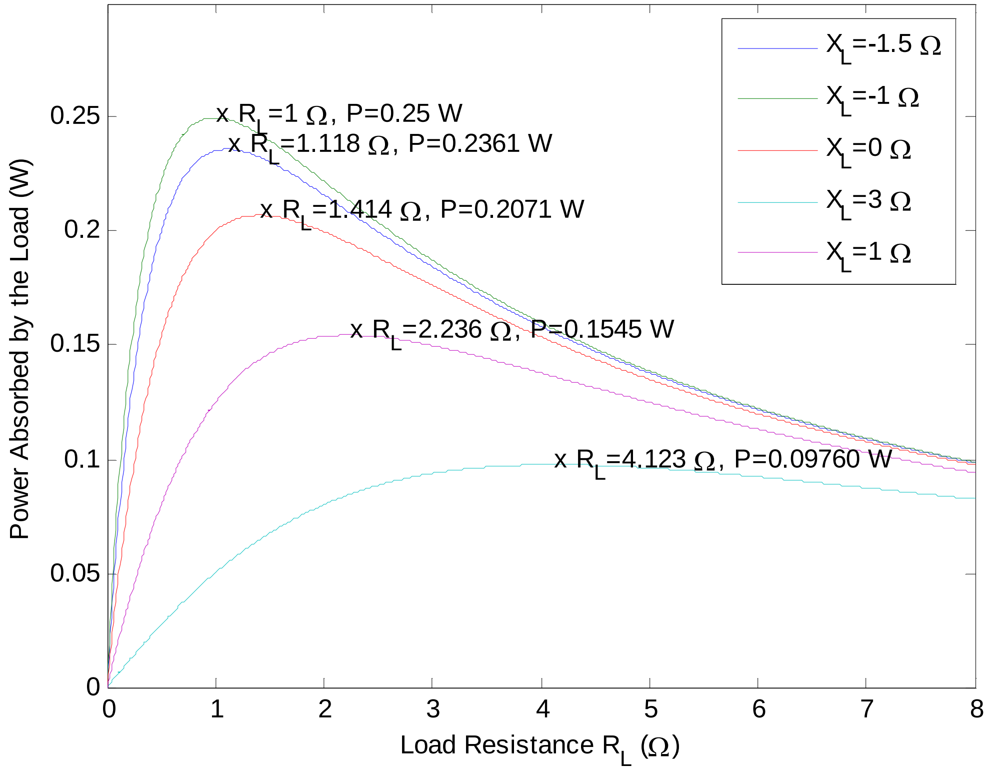

Fig. 6. Graph of the power absorbed by the load as the load resistance is varied, for various fixed values of the load reactance. The maximum power point for each curve is labeled. The internal impedance of the source is $Z = 1 + j1\ \Omega$ and $\boldsymbol{E} = 1\angle 0$ V.

C. Maximum Power Transfer Theorem when Load Impedance Angle cannot be Varied, but the Load Impedance Magnitude can be Chosen

In this third case, we are free to choose the optimum value of the magnitude of the impedance, $|Z_L|$, but the value of the angle of the load impedance cannot be changed. To find the optimum value of $|Z_L|$, we write (14) as

$$P_L = \frac{|Z_L|\cos\theta_L}{\left(R + |Z_L|\cos\theta_L\right)^2 + \left(X + |Z_L|\sin\theta_L\right)^2} E^2, \tag{20}$$

where we have used the fact that $R_L = |Z_L|\cos\theta_L$ and $X_L = |Z_L|\sin\theta_L$.

Expanding the brackets in (20) gives

$$P_L = \frac{|Z_L|}{|Z_L|^2 + 2|Z_L|\left(R\cos\theta_L + X\sin\theta_L\right) + R^2 + X^2} E^2\cos\theta_L. \qquad (21)$$

Dividing the numerator and denominator of (21) by $R^2 + X^2$ gives

$$P_L = \frac{\dfrac{|Z_L|}{R^2+X^2}}{\dfrac{|Z_L|^2}{R^2+X^2} + \dfrac{2|Z_L|\left(R\cos\theta_L + X\sin\theta_L\right)}{R^2+X^2} + 1} E^2\cos\theta_L. \qquad (22)$$

However, we can rewrite (22) as

$$P_L = \frac{\dfrac{|Z_L|}{\sqrt{R^2+X^2}}}{\dfrac{|Z_L|^2}{R^2+X^2} + \dfrac{|Z_L|}{\sqrt{R^2+X^2}}\dfrac{2\left(R\cos\theta_L + X\sin\theta_L\right)}{\sqrt{R^2+X^2}} + 1} \frac{E^2\cos\theta_L}{\sqrt{R^2+X^2}}. \qquad (23)$$

Defining $k = \dfrac{|Z_L|}{\sqrt{R^2+X^2}}$, (23) becomes

$$\begin{aligned} P_L &= \frac{k}{k^2 + \dfrac{2\left(R\cos\theta_L + X\sin\theta_L\right)}{\sqrt{R^2+X^2}}k + 1} \frac{E^2\cos\theta_L}{\sqrt{R^2+X^2}} \\ &= \frac{1}{k + k^{-1} + \dfrac{2\left(R\cos\theta_L + X\sin\theta_L\right)}{\sqrt{R^2+X^2}}} \frac{E^2\cos\theta_L}{\sqrt{R^2+X^2}}. \end{aligned} \qquad (24)$$

Clearly, in order for (24) to be a maximum, $k + k^{-1}$ must be a minimum. However, we have already established that $k = 1$ accomplishes this. Hence, for maximum power transfer to the load, $k = \dfrac{|Z_L|}{\sqrt{R^2+X^2}} = 1$, or $|Z_L| = \sqrt{R^2+X^2}$, which of course, is the same result obtained with calculus. Hence, the magnitude of the load impedance should be equal to the magnitude of the internal impedance.

Additionally, we can obtain the maximum power that is transferred to the load from (24) and the fact that $k = 1$. Hence,

$$P_{L\max} = \frac{1}{2+\frac{2\left(R\cos\theta_L + X\sin\theta_L\right)}{\sqrt{R^2+X^2}}}\frac{E^2\cos\theta_L}{\sqrt{R^2+X^2}}$$

$$= \frac{E^2}{4\left[\frac{\sqrt{R^2+X^2}+R\cos\theta_L + X\sin\theta_L}{2\cos\theta_L}\right]}. \quad (25)$$

Note that if the load is purely resistive, i.e. $\theta_L = 0$, and if the internal reactance is zero, (25) reduces to (6) as it should.

Furthermore, if the load angle is equal to the negative of the angle of the internal impedance, i.e. $\theta_L = -\theta$, where $\theta = \tan^{-1}\frac{X}{R}$, then the load will be optimum, since the load would be the conjugate of the internal impedance. Hence, (25) should reduce to (6) for this case as well.

To see that this happens, recall that $R = \sqrt{R^2+X^2}\cos\theta$ and $X = \sqrt{R^2+X^2}\sin\theta$. However, $\theta = -\theta_L$ by assumption. Hence, substituting this into the two previous equations gives $\cos\theta_L = \frac{R}{\sqrt{R^2+X^2}}$ and $\sin\theta_L = \frac{-X}{\sqrt{R^2+X^2}}$. Finally, we put these into (25) to get

$$P_{L\max} = \frac{E^2}{4\left[\frac{\sqrt{R^2+X^2}+R\frac{R}{\sqrt{R^2+X^2}}+X\frac{-X}{\sqrt{R^2+X^2}}}{2\frac{R}{\sqrt{R^2+X^2}}}\right]}. \quad (26)$$

Multiplying the numerator and denominator of the lower fraction in (26) by $\sqrt{R^2+X^2}$ gives the desired result, i.e.

$$P_{L\max} = \frac{E^2}{4\left[\frac{R^2+X^2+R^2-X^2}{2R}\right]}$$

$$= \frac{E^2}{4R}. \quad (27)$$

Summary 5: If the phase of the load cannot be varied, the magnitude of the load impedance should be chosen to be equal to the magnitude of the internal impedance, i.e. $|Z_L| = \sqrt{R^2 + X^2}$**, for maximum power transfer to the load. Furthermore, the actual maximum power transferred to the load will then be given by (25), where** E **is now the *rms* magnitude of the voltage source *E*.**

To illustrate this, we use (20) to plot the absorbed power as a function of the magnitude of the load impedance, with the load phase being a fixed value. This plot is given in Fig. 7, where we have assumed that the internal impedance of the source is $Z = 1 + j1\Omega$ and the applied voltage is $\boldsymbol{E} = 1\angle 0$ V. Clearly, the optimum value of the magnitude of the load impedance is well-predicted by $|Z_L| = \sqrt{R^2 + X^2}$, whereas the maximum power absorbed is well-predicted by (25). Furthermore, when the load phase is its optimum value, i.e. -45^o, the absorbed power is at its maximum value of 0.25 W, as it should be.

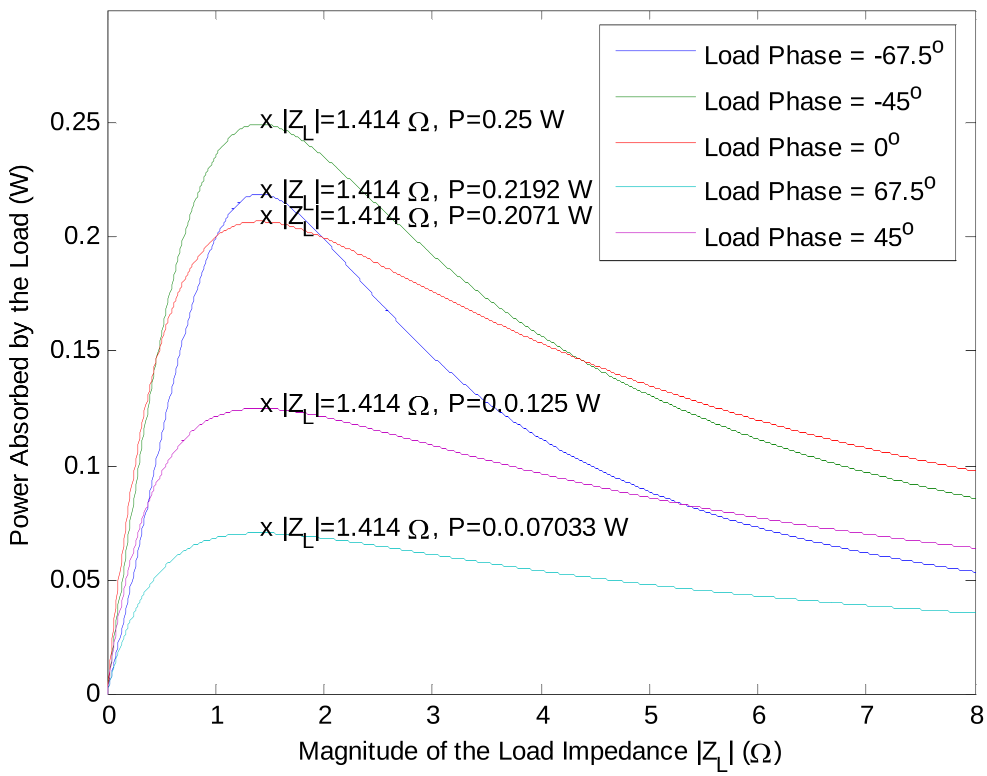

Fig. 7. Graph of the power absorbed by the load as the magnitude of the load impedance is varied, for various fixed values of the phase. The maximum power point for each curve is labeled. The internal impedance of the source is $Z = 1 + j1\,\Omega$ and $\boldsymbol{E} = 1\angle 0$ V.

III. Conclusion

In this paper, the maximum power transfer theorem has been derived without the use of calculus: the method requires the student to know only algebra. The theorem for DC circuits was derived for two cases: (i) determination of the optimum value of the load resistor for maximum power transfer to it, and (ii) determination of the optimum value of the internal resistance of the source, for maximum power transfer to the load. On the other hand, proofs for AC circuits were given for three cases: (i) when both the load resistor and the load reactance can be freely chosen, (ii) when only the load resistor can be chosen because the load reactance is fixed, and (iii) when the magnitude of the load impedance can be chosen but the angle of the load impedance is fixed.

IV. Appendix

In this appendix we review the method of Paul and Gardner [1].

From (1),

$$\begin{aligned} V_L &= \frac{R_L}{R_L + R} E \\ &= \frac{2R_L}{R_L + R} \frac{E}{2} \\ &= \left(\frac{R_L + R + R_L - R}{R_L + R} \right) \frac{E}{2} \\ &= \left(\frac{R_L + R}{R_L + R} + \frac{R_L - R}{R_L + R} \right) \frac{E}{2} \\ &= \left(1 + \frac{R_L - R}{R_L + R} \right) \frac{E}{2}. \end{aligned} \tag{A1}$$

Also, the current through the load is

$$\begin{aligned} I_L &= \frac{V_L}{R_L} = \frac{1}{R_L + R}E \\ &= \frac{2R}{R_L + R}\frac{E}{2R} \\ &= \left(\frac{R_L + R - (R_L - R)}{R_L + R}\right)\frac{E}{2R} \\ &= \left(\frac{R_L + R}{R_L + R} - \frac{R_L - R}{R_L + R}\right)\frac{E}{2R} \\ &= \left(1 - \frac{R_L - R}{R_L + R}\right)\frac{E}{2R}. \end{aligned} \tag{A2}$$

Hence, the power absorbed by the load is

$$\begin{aligned} P_L &= V_L I_L \\ &= \left(1 + \frac{R_L - R}{R_L + R}\right)\frac{E}{2}\left(1 - \frac{R_L - R}{R_L + R}\right)\frac{E}{2R} \\ &= \left(1 - \left(\frac{R_L - R}{R_L + R}\right)^2\right)\frac{E^2}{4R}. \end{aligned} \tag{A3}$$

Clearly, the power is a maximum when $\left(\frac{R_L - R}{R_L + R}\right)^2$ is a minimum, which of course is true for $R_L = R$.

It is also interesting to note that Paul and Gardner also interpret (A3) from the reflection coefficient of a transmission line point of view. Please see [1] for details.

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