DETERMINING THE EFFECTIVE OR RMS VOLTAGE OF VARIOUS WAVEFORMS WITHOUT CALCULUS

by

Kenneth V. Cartwright, Ph.D.
kvc@batelnet.bs
School of Sciences and Technology
College of The Bahamas

Abstract: The *rms* (root-mean-square) value of various waveforms is determined without the use of calculus, which should be of benefit to the Technology student. These waveforms include the half-wave rectified sinusoid, the quarter-wave rectified sinusoid, a waveform consisting of a dc voltage added to a sinusoid, the sum of two sinusoids, which may be periodic or aperiodic, a triangular waveform, and a pulse train. Furthermore, in order to compute the *rms* value, the mean of the square of a voltage must be determined over an observation time. A discussion of the importance of using the correct observation time is also provided.

I. Introduction

Several authors, e.g. [1] and [2], have shown that the effective or root-mean-square (*rms*) voltage of a sinusoid can be obtained without the use of calculus. In this paper, we show that this can also be done for other voltage waveforms, although this does not seem to be widely recognized.

These methods all use the definition of the *rms* voltage of a waveform $v(t)$, which is found simply by squaring that waveform, taking the mean (or average) of that squared waveform and then computing the square root, as given in equation (1):

$$V_{rms} = \sqrt{\text{mean}[v(t)*v(t)]} = \sqrt{\frac{\text{area under the curve of }[v(t)*v(t)]}{\text{observation length}}}\ , \qquad (1)$$

where the observation length is the period (or integer multiple of it) for periodic signals, and is as long as possible for aperiodic signals. More will be said about this in Section II C.

This definition is, of course, a purely mathematical one. The physical interpretation of the *rms* value has already been covered by many authors, including Rico [1]. Hence, in this paper, we will be considering (1), without any attempt to repeat its physical interpretation, which remains the same as it was for the purely sinusoidal case.

II. Analytical Determination of the RMS Value of Some Selected Waveforms

In this section, we will show how the *rms* value of the following waveforms can be determined without calculus:

(i) Half-wave rectified sinusoid
(ii) Quarter-wave rectified sinusoid
(iii) A DC voltage added to a sinusoid
(iv) The sum of two sinusoids, when the sum waveform is periodic or aperiodic
(v) A triangular waveform, and
(vi) A pulse train.

A. RMS Value of a Half-wave Rectified Sinusoid

A graph of a positive half-wave rectified sinusoid is shown in Fig. 1. For this waveform, we give two different methods for finding its *rms* value. In Method I, the half-wave rectified waveform is written as the sum of a sinusoid and a full-wave rectified sinusoid. In Method II, we use a piecewise mathematical equation for the waveform.

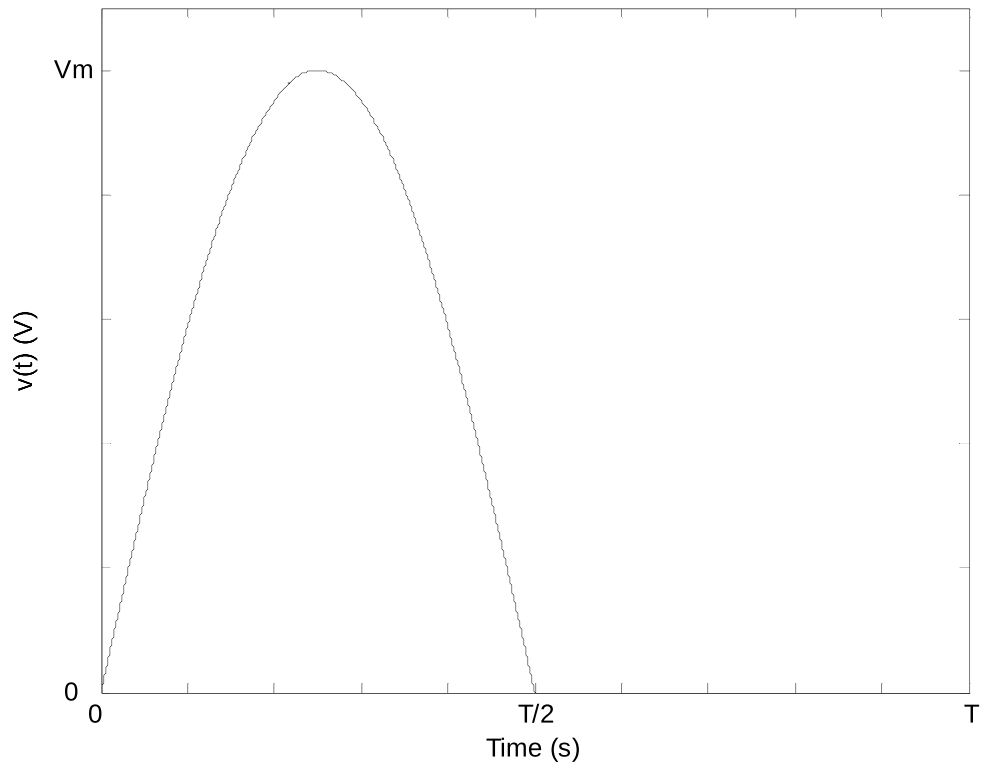

Fig. 1. Graph of a positive half-wave rectified sine-wave.

Method I

The equation for a positive half-wave rectified sinusoid can be written as

$$v(t) = V_m \sin(\omega t + \theta)/2 + \left|V_m \sin(\omega t + \theta)\right|/2. \tag{2}$$

Note that the second term in (2) is simply a full-wave rectified sinusoid.

Squaring (2) gives

$v(t) * v(t) = V_m^2 \sin^2(\omega t + \theta)/4 + \left|V_m \sin(\omega t + \theta)\right|^2/4 + 2V_m \sin(\omega t + \theta)\left|V_m \sin(\omega t + \theta)\right|$, which can also be written as

$$v^2(t) = V_m^2 \sin^2(\omega t + \theta)/2 + 2V_m \sin(\omega t + \theta)\left|V_m \sin(\omega t + \theta)\right|. \tag{3}$$

Now we need to take the mean of (3), which consists of the sum of two terms. Fortunately, the mean of a sum is the sum of the means, i.e.

$\text{mean}(a+b) = \text{mean}(a) + \text{mean}(b)$. In this case, $a = V_m^2 \sin^2(\omega t + \theta)$ and $b = 2V_m \sin(\omega t + \theta)\left|V_m \sin(\omega t + \theta)\right|$.

Note the mean of the second term in (3) is zero, i.e. $\text{mean}(b) = 0$. To see this, we sketch b as a function of the angle $(\omega t + \theta)$, as shown in Fig.2.

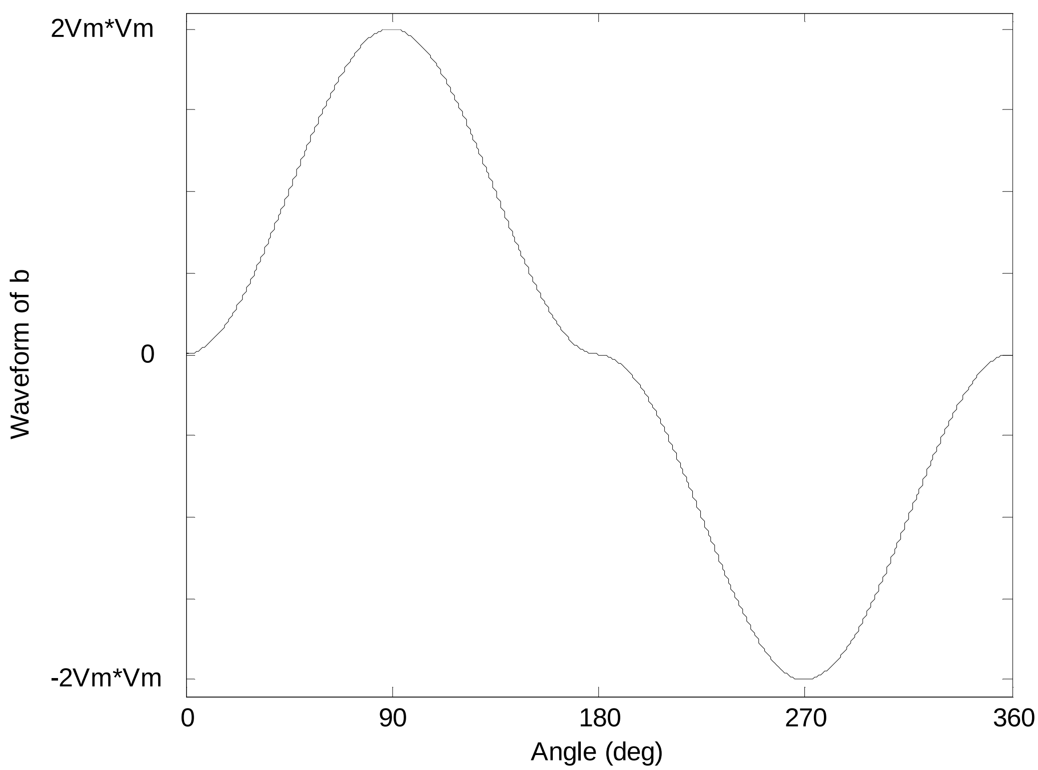

Fig. 2. Graph of *b*, the second term in (3). This clearly shows that the mean(*b*) = 0.

Note that for every positive value of b in Fig. 2, there is a corresponding negative value. Hence, the average over a complete cycle is zero.

(Or, by symmetry, it is clear that the negative area under the curve for the second half-cycle has the same magnitude as the positive area under the curve for the first half cycle. Therefore, the total area over a complete cycle is zero, and because the mean is simply the area over a complete cycle, divided by the period, the mean has to be zero).

On the other hand, the mean of *a* is given by

$$\text{mean}(V_m^2 \sin^2(\omega t + \theta)/2) = \text{mean}(V_m^2/4 - V_m^2/4\cos(2\omega t + 2\theta)), \tag{4}$$

where we have used the well-known trigonometric identity

$$\sin^2(x) = (1-\cos(2x))/2\,. \tag{5}$$

Equation (4) further simplifies to

$$\text{mean}(a) = \text{mean}(V_m^2/4) - \text{mean}(V_m^2/4\cos(2\omega t + 2\theta)). \tag{6}$$

The first term on the right-hand-side of (6) is simply the mean of a constant which of course is just that constant. On the other hand, the second term on the right-hand-side of (6) is the mean of a sinusoid, which is well-known to be zero, as pointed out by Rico [1]. Hence, $\text{mean}(a) = V_m^2/4$, which then of course makes the mean of (3) also this value.

Finally, we need to take the square root of the mean of (3) to arrive at our conclusion: the *rms* value of a positive half-wave rectified sinusoid is $\sqrt{V_m^2/4} = V_m/2$, which is also the answer given by calculus.

In the above, we assumed that we had a positive half-wave rectified sinusoid. For a negative half-wave rectified sinusoid, the proof is similar, except its equation is given by $v(t) = V_m\sin(\omega t+\theta)/2 - |V_m\sin(\omega t+\theta)|/2.$

Method II

The equation for the positive half-wave rectified sine-wave shown in Fig. 1 can also be written in piecewise fashion as

$$\begin{aligned} v(t) &= V_m \sin\left(\frac{2\pi}{T}t\right) && \text{for } 0 \le t \le \frac{T}{2} \\ &= 0 && \text{for } \frac{T}{2} \le t \le T. \end{aligned} \tag{7}$$

Squaring (7) gives

$$\begin{aligned} v^2(t) &= V_m^2 \sin^2\left(\frac{2\pi}{T}t\right) && \text{for } 0 \le t \le \frac{T}{2} \\ &= 0 && \text{for } \frac{T}{2} \le t \le T. \end{aligned} \tag{8}$$

A graph of this squared waveform is shown in Fig. 3. Furthermore, using (5), (8) can be written as a constant minus a cosine wave, as given below in (9).

$$
\begin{aligned}
v^2(t) &= \frac{V_m^2}{2} - \frac{V_m^2}{2}\cos\left(\frac{4\pi}{T}t\right) && \text{for } 0 \le t \le \frac{T}{2} \\
&= 0 && \text{for } \frac{T}{2} \le t \le T.
\end{aligned}
\tag{9}
$$

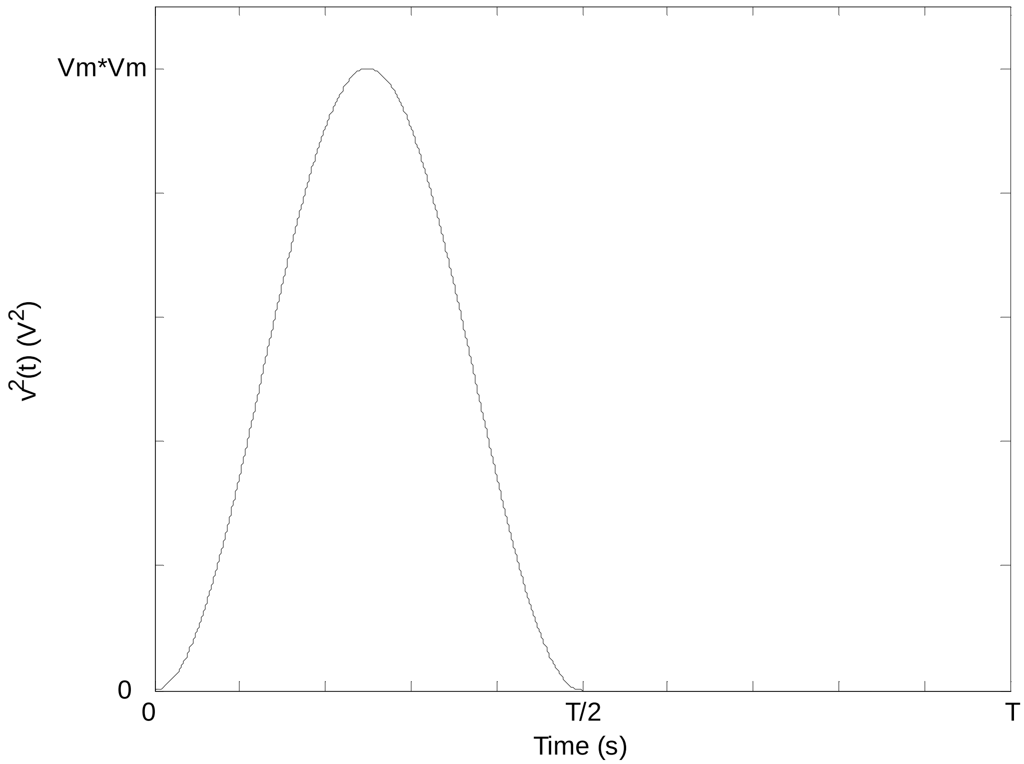

Fig. 3. Plot of the square of the half-wave rectified sine-wave of Fig.1.

Inspection of (9) shows that the squared waveform can be written as the difference of two waveforms, i.e.,

$$v^2(t) = a(t) - b(t)\,, \tag{10}$$

where

$$
\begin{aligned}
a(t) &= \frac{V_m^2}{2} && \text{for } 0 \le t \le \frac{T}{2} \\
&= 0 && \text{for } \frac{T}{2} \le t \le T
\end{aligned}
$$

and

$$b(t) = \frac{V_m^2}{2}\cos\left(\frac{4\pi}{T}t\right) \qquad \text{for } 0 \le t \le \frac{T}{2}$$

$$= 0 \qquad \text{for } \frac{T}{2} \le t \le T.$$

These latter two waveforms are shown in Fig. 4(a) and Fig. 4(b).

Now we need to find the mean of (10), which is

$$\text{mean}\left(v^2(t)\right) = \text{mean}\left(a^2(t) - b^2(t)\right) = \text{mean}\left(a^2(t)\right) - \text{mean}\left(b^2(t)\right). \qquad (11)$$

Fortunately, the means on the right hand side of (11) are easily determined. Using the fact that the mean is simply the area under the curve over a period, divided by the period, as given in (1), we find $\text{mean}\left(a^2(t)\right) = V_m^2/4$ and $\text{mean}\left(b^2(t)\right) = 0$. Hence, from (11), $\text{mean}\left(v^2(t)\right) = V_m^2/4$ and so the *rms* value of the waveform in Fig. 1 is $V_m/2$, as determined earlier with Method I.

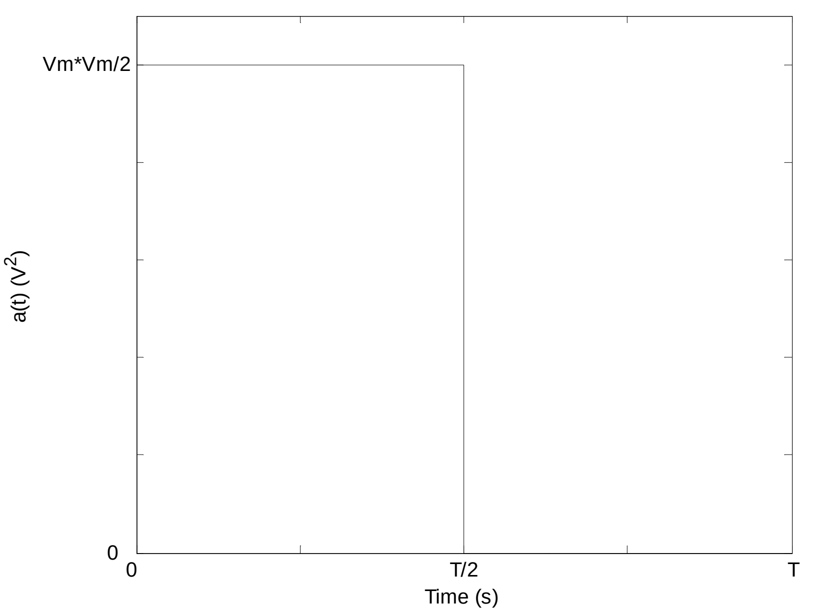

Fig. 4(a). Piecewise constant component of the square of a half-wave rectified sine-wave. Note that the mean of this waveform over a complete period is $V_m^2/4$.

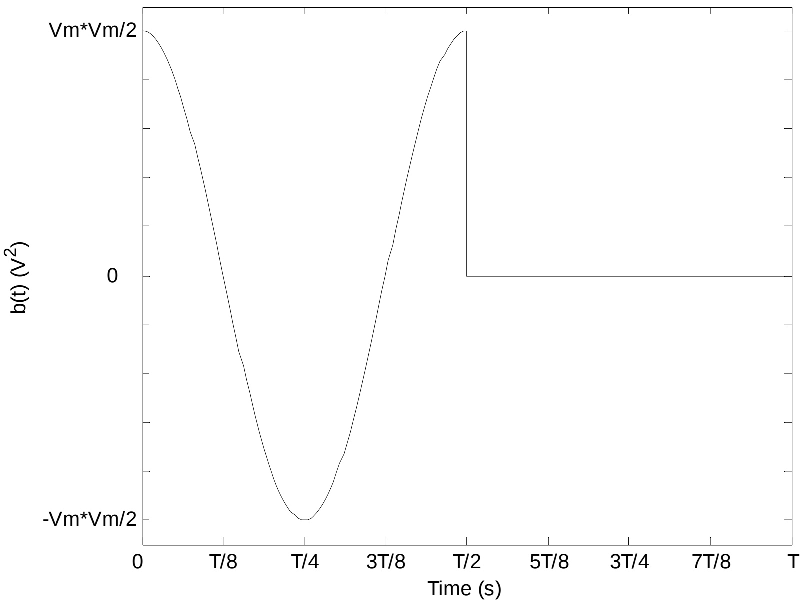

Fig. 4(b). Piecewise sinusoidal component of the square of a half-wave rectified sine-wave. Note that the mean of this waveform over a complete period is zero.

B. RMS Value of a Quarter-Wave Rectified Sine-wave

The quarter-wave rectified sine-wave shown in Fig. 5 is typically found in silicon controlled rectifier (SCR) circuits. Fortunately, the *rms* value of this waveform can be found with Method II, as described above in Section II-A.

Indeed, the equation for the quarter-wave rectified sine-wave shown in Fig. 5 can be written in piecewise fashion as

$$\begin{aligned} v(t) &= V_m \sin\left(\frac{2\pi}{T}t\right) && \text{for } \frac{T}{4} \le t \le \frac{T}{2} \\ &= 0 && \text{for } 0 \le t \le \frac{T}{4} \text{ and } \frac{T}{2} \le t \le T. \end{aligned} \tag{12}$$

Squaring (12) gives

$$\begin{aligned} v^2(t) &= V_m^2 \sin^2\left(\frac{2\pi}{T}t\right) && \text{for } \frac{T}{4} \le t \le \frac{T}{2} \\ &= 0 && \text{for } 0 \le t \le \frac{T}{4} \text{ and } \frac{T}{2} \le t \le T. \end{aligned} \tag{13}$$

A graph of this squared waveform is shown in Fig. 6. Furthermore, using (5), (13) can be written as a constant minus a cosine wave, as given below in (14).

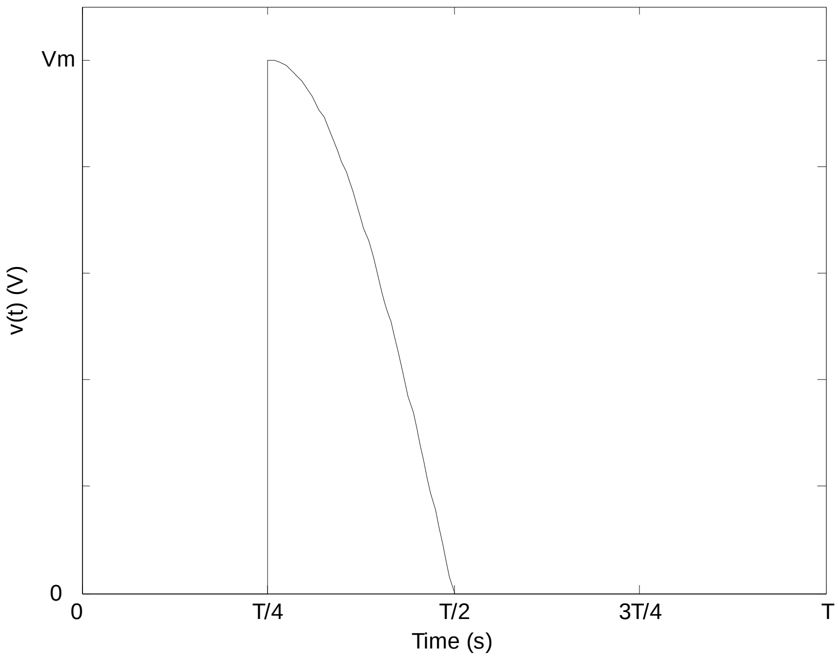

Fig. 5. Graph of a positive quarter-wave rectified sine-wave, which is useful in silicon-controlled rectifier circuits.

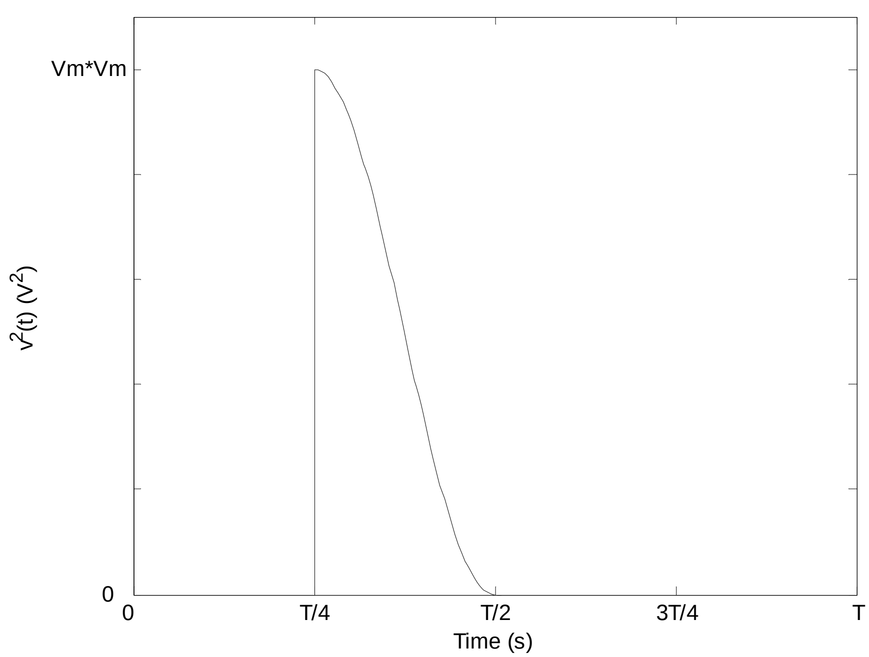

Fig. 6. Graph of the square of the quarter-wave rectified sine-wave.

$$v^2(t) = \frac{V_m^2}{2} - \frac{V_m^2}{2}\cos\left(\frac{4\pi}{T}t\right) \qquad \text{for } \frac{T}{4} \le t \le \frac{T}{2}$$
$$= 0 \qquad \text{for } 0 \le t \le \frac{T}{4} \text{ and } \frac{T}{2} \le t \le T. \tag{14}$$

As noted before, (14) can be written as the difference of two waveforms, i.e.,

$$v^2(t) = a(t) - b(t)\ , \tag{15}$$

where

$$a(t) = \frac{V_m^2}{2} \qquad \text{for } \frac{T}{4} \le t \le \frac{T}{2}$$
$$= 0 \qquad \text{for } 0 \le t \le \frac{T}{4} \text{ and } \frac{T}{2} \le t \le T$$

and

$$b(t) = \frac{V_m^2}{2}\cos\left(\frac{4\pi}{T}t\right) \qquad \text{for } \frac{T}{4} \le t \le \frac{T}{2}$$
$$= 0 \qquad \text{for } 0 \le t \le \frac{T}{4} \text{ and } \frac{T}{2} \le t \le T.$$

These latter two waveforms are shown in Fig. 7(a) and Fig. 7(b).

Now, using (11), we need to find the mean of (15). Fortunately, the means on the right hand side of (11) are easily determined to be $\text{mean}\left(a^2(t)\right) = V_m^2/8$ and $\text{mean}\left(b^2(t)\right) = 0$. Hence, $\text{mean}\left(v^2(t)\right) = V_m^2/8$ and so the *rms* value of the waveform in Fig. 5 is $\frac{V_m}{2\sqrt{2}}$, as can also be verified with calculus.

We wish to point out that the *rms* value of other waveforms that are found in SCR or Triac circuits can be found using Method II, e.g., the waveform shown in Fig. 8.

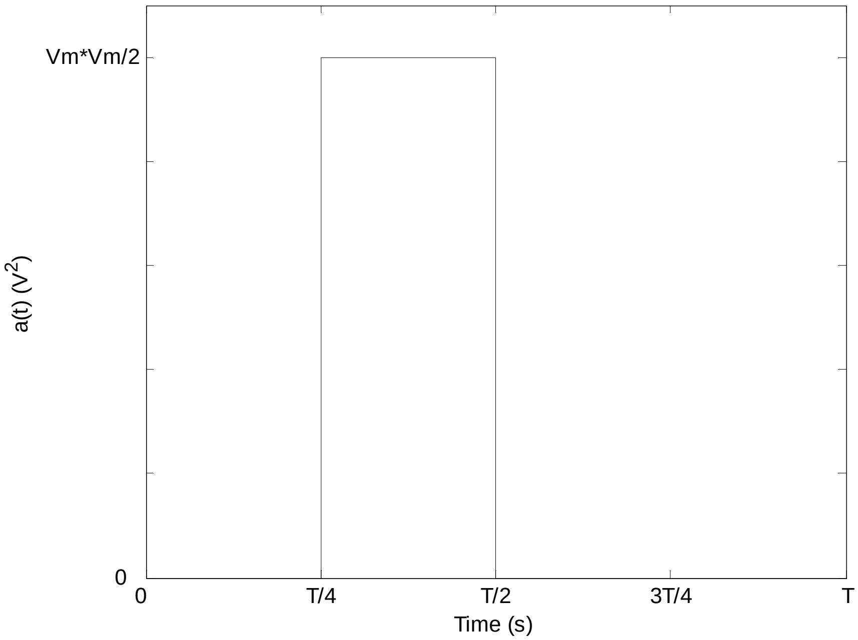

Fig. 7(a). Piecewise constant component of the square of a quarter-wave rectified sine-wave. Note that the mean of this waveform over a complete period is $V_m^2/8$.

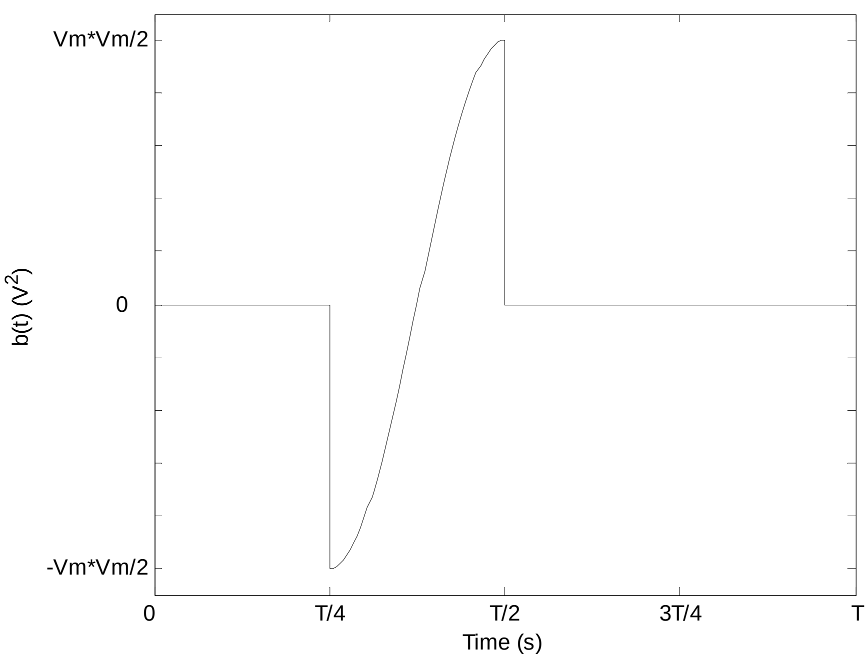

Fig. 7(b). Piecewise sinusoidal component of the square of a quarter-wave rectified sine-wave. Note that the mean of this waveform over a complete period is zero.

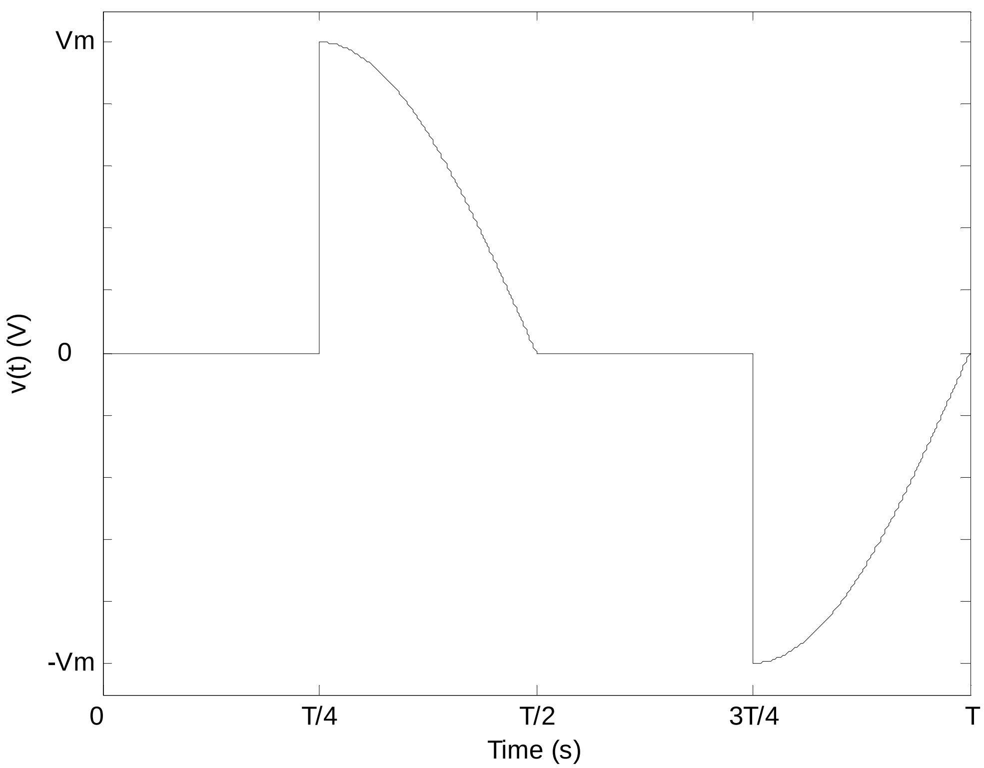

Fig. 8. Another waveform that can be found in electronic circuits, and whose *rms* value can be found with Method II.

C. RMS Value of DC and a Sinusoid

We now turn our attention to the *rms* value of a waveform that is the sum of a DC voltage and a sinusoid. In some Technology textbooks (e.g. [2]), this result is just given without any justification. In this subsection, we show how we can derive this result without calculus.

In this case, $v(t) = V_{dc} + V_m \sin(\omega t + \theta),$ where V_{dc} is the value of the DC voltage and V_m is the amplitude of the sinusoid. Squaring this gives $V_{dc}^2 + 2V_{dc}V_m \sin(\omega t + \theta) + V_m^2 \sin^2(\omega t + \theta).$ Taking the mean produces the mean of three terms. The mean of the first term is simply $V_{dc}^2.$ The mean of the second term is zero since we are taking the mean of a sinusoid. The mean of the third term is $\text{mean}(V_m^2/2 - V_m^2/2\cos(2\omega t + 2\theta)),$ where we have again used (5). Hence, the mean of this third term becomes $V_m^2/2.$ Therefore,

$$\text{mean}(V_{dc}^2 + 2V_{dc}V_m \sin(\omega t + \theta) + V_m^2 \sin^2(\omega t + \theta)) = V_{dc}^2 + V_m^2/2. \tag{16}$$

Taking the square root of (16) gives our desired result, i.e. the *rms* value of a DC voltage plus a sinusoid is given by $V_{rms} = \sqrt{V_{dc}^2 + V_m^2 / 2}.$

D. RMS Value of the Sum of Two Sinusoids

In this subsection, we find the *rms* value of the sum of two sinusoids, i.e. we assume that $v(t) = v_1(t) + v_2(t) = V_{m1}\sin(\omega_1 t + \theta_1) + V_{m2}\sin(\omega_2 t + \theta_2).$ First, we square this waveform to get $v^2(t) = V_{m1}^2 \sin^2(\omega_1 t + \theta_1) + V_{m1}^2 \sin^2(\omega_2 t + \theta_2) + 2V_{m1}V_{m2}\sin(\omega_1 t + \theta)\sin(\omega_2 t + \theta_2).$ Using (5) and the trigonometric identity $\sin(x)\sin(y) = \cos(x-y)/2 - \cos(x+y)/2$, we can write

$$\begin{aligned} v^2(t) = {} & V_{m1}^2/2 - V_{m1}^2/2\cos(2\omega_1 t + 2\theta_1) + V_{m2}^2/2 - V_{m2}^2/2\cos(2\omega_2 t + 2\theta_2) \\ & + V_{m1}V_{m2}\cos[(\omega_1 - \omega_2)t + \theta_1 - \theta_2] + V_{m1}V_{m2}\cos[(\omega_1 + \omega_2)t + \theta_1 + \theta_2)]. \end{aligned} \tag{17}$$

Inspection of (17) shows that the square consists of $v^2(t) = V_{m1}^2/2 + V_{m2}^2/2 + \text{sum}$ of sinusoids. Hence, when we ***properly*** take the mean of (17), the mean of the sum of sinusoids will be zero and so the *rms* value will be $V_{rms} = \sqrt{V_{m1}^2/2 + V_{m2}^2/2}.$

By properly taking the mean of (17), we mean that the observation length (time) over which the mean of $v^2(t)$ is computed must be carefully considered. There are three different cases to consider: (i) one sinusoid is an harmonic of the other, i.e. $\omega_2 = N\omega_1$, where N is a positive integer, (ii) the sinusoids are not harmonically related, but nonetheless the sum is periodic, and (iii) the sum of the sinusoids is not periodic.

In the first case, the observation time is simply the period of the sinusoid with the lowest frequency.

In the second case, the observation time is the period (or integer multiple) of the *sum* of the sinusoids contained in $v^2(t)$ and not the period of each individual sinusoid. The reader should recall that the period of the sum of sinusoids might be quite longer than the period of the individual sinusoid that has the longest period. For example, suppose $v(t) = \sin(3\pi t) + \sin(6\pi/5t)$; so that $v^2(t)$ has frequencies $2\omega_1 = 6\pi$ rad/s, $2\omega_2 = 12\pi/5$ rad/s, $\omega_1 - \omega_2 = 9\pi/5$ rad/s and $\omega_1 + \omega_2 = 21\pi/5$ rad/s. The corresponding period of each of these is $T_1 = 1/3$ s, $T_2 = 5/6$ s, $T_3 = 10/9$ s and $T_4 = 10/21$ s, respectfully. Hence, in this case, the period of the sum is given by $T_{sum} = 10/3$ s, as this is the minimum time necessary to ensure that ***all*** the sinusoids have gone through complete cycles. Indeed, for this example, $T_{sum} = 10/3s = 10T_1 = 4T_2 = 3T_3 = 7T_4$: that is, the first sinusoid completes ten cycles, the second completes four, the third sinusoid completes three, and the fourth completes seven cycles.

Fig. 9(a) shows a PSpice plot of $v(t) = \sin(3\pi t) + \sin(6\pi / 5t)$, and Fig. 9(b) shows the square of that same waveform. Clearly, the period of $v^2(t)$ is $T_{sum} = 10/3$ s, as expected. Therefore, the time over which the mean of $v^2(t)$ is computed should be integer multiples of 10/3 s. This fact is verified in Fig. 9(c), which shows a PSpice plot of the mean of $v^2(t)$ as a function of observation length (time). From the graph, it is clearly seen that PSpice rightly determines the correct mean value of unity at integer multiples of 10/3 s, and also at other observations lengths. However, to determine these latter observation lengths requires calculus: hence, we will limit ourselves to calculating the mean over the period of the sum, or integer multiples of this.

Finally, in the third case, the period of the sum of two sinusoids might be infinite and so the mean computation has to be taken at specific values of time, or can only be well-approximated by computing the mean over a "long" period of time. For example, see Fig. 10(a) which shows a PSpice plot of $v(t) = \sin(2\pi t) + \sin(2\sqrt{2}\pi t)$. Note that this waveform is not periodic because the ratio of the frequencies of its sine waves is an irrational number. Also, see Fig. 10(b) which shows the square of this waveform, which also is not periodic. Furthermore, Fig. 10(c) shows the mean of the squared waveform as a function of observation length (time). Clearly, the correct determination of the mean value of unity for this aperiodic waveform depends upon the mean being taken over specific times. However, to calculate these specific times requires the use of calculus. On the other hand, as can be seen from Fig. 10(c), the longer the observation time to calculate the mean, the more accurate the calculation becomes. Therefore, we need not worry about determining the exact times over which the mean should be calculated; rather, we just allow the observation time to be a lot longer than the period of the lowest frequency sine wave in the squared waveform.

We end this subsection by pointing out that this method of finding the *rms* value of a waveform which is the sum of two sinusoids can easily be generalized to a waveform which is the sum of *N* sinusoids, where *N* is any positive integer.

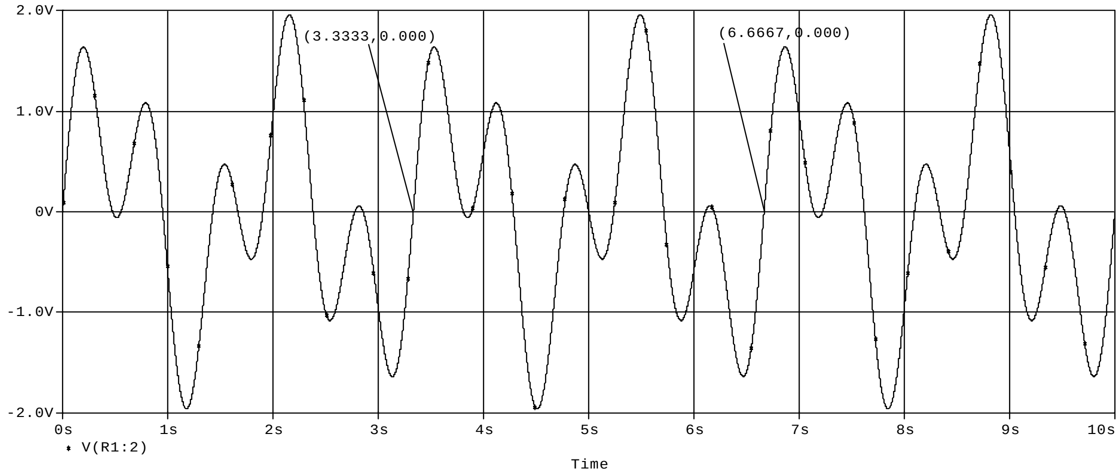

Fig. 9(a). Plot of $v(t) = \sin(3\pi t) + \sin(6\pi t/5)$. Note that the period is 10/3 s.

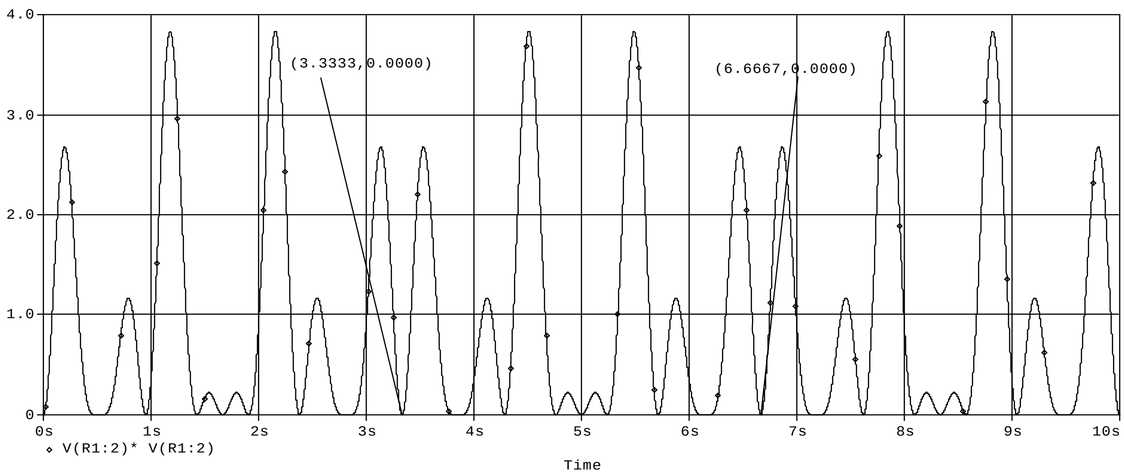

Fig. 9(b). Plot of $v^2(t) = \left(\sin(3\pi t) + \sin(6\pi t/5)\right)^2$. Note that the period is 10/3 s.

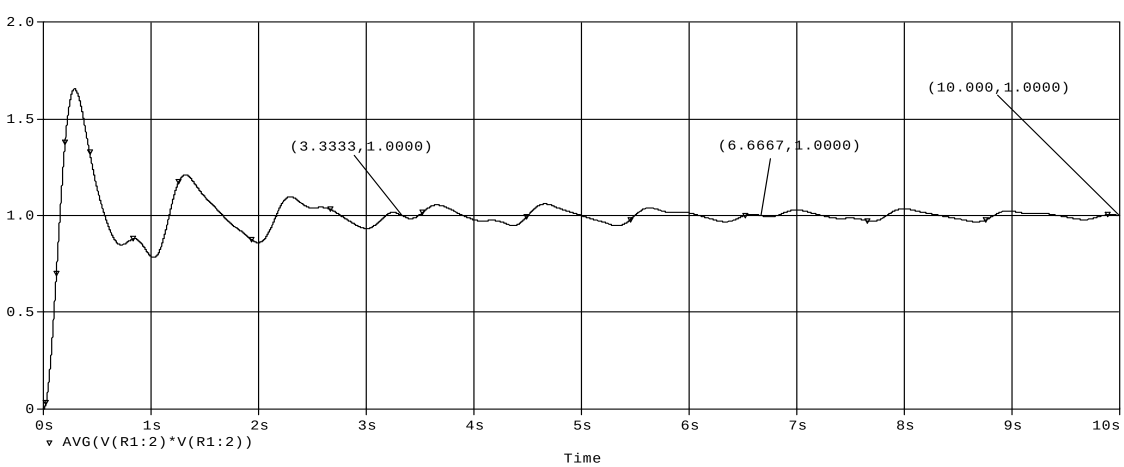

Fig. 9(c). Plot of $\text{mean}\left[v^2(t)\right] = \text{mean}\left[\left(\sin(3\pi t) + \sin(6\pi t/5)\right)^2\right]$ as a function of observation length (time) of the mean calculation.

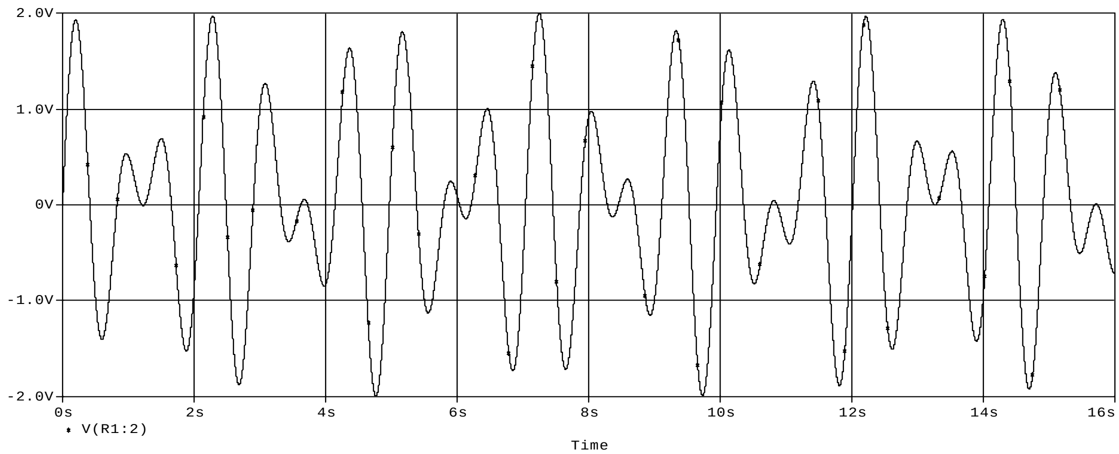

Fig. 10(a). Plot of $v(t) = \sin(2\pi t) + \sin(2\sqrt{2}\pi t)$. Note the lack of periodicity.

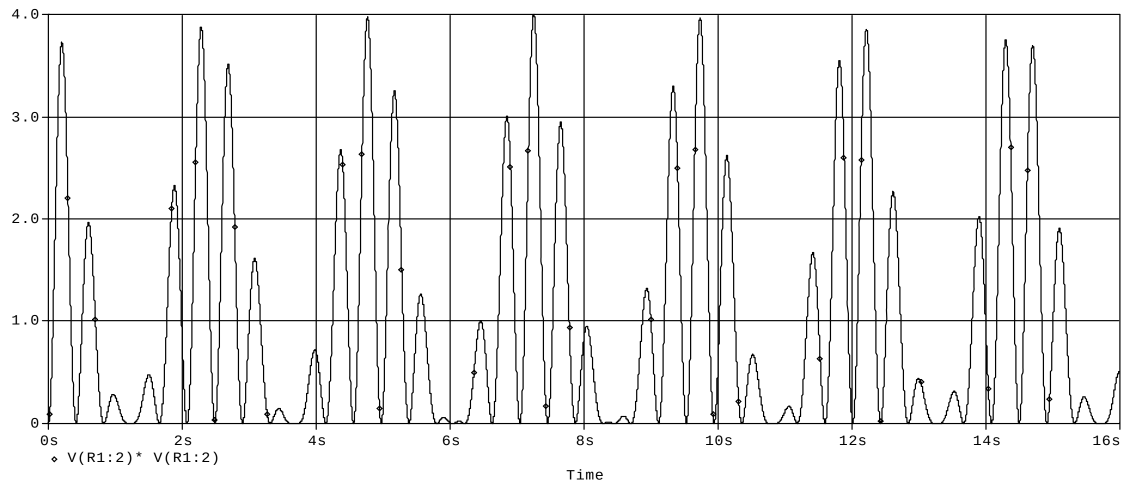

Fig. 10(b). Plot of $v^2(t) = \left(\sin(2\pi t) + \sin(2\sqrt{2}\pi t)\right)^2$. Note the lack of periodicity.

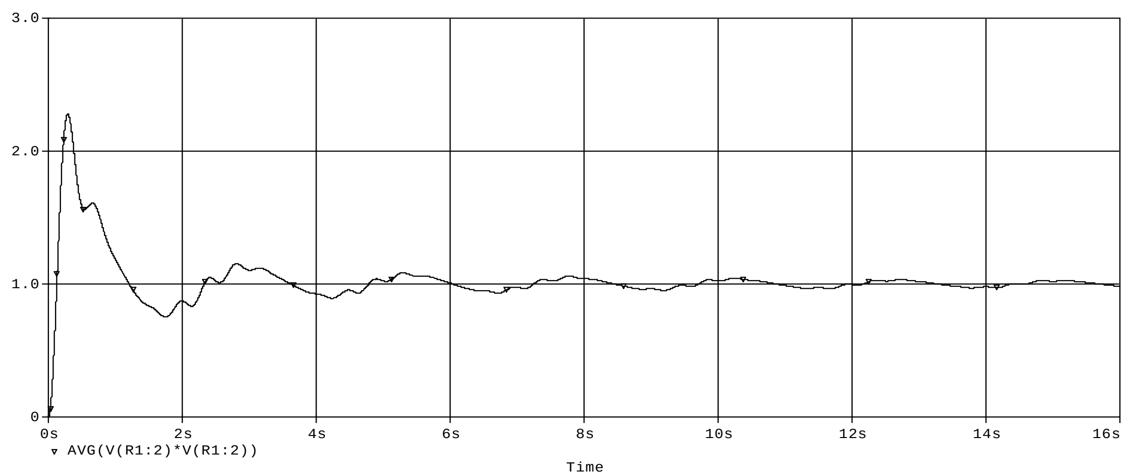

Fig. 10(c). Plot of $\text{mean}\left[v^2(t)\right] = \text{mean}\left[\left(\sin(2\pi t) + \sin(2\sqrt{2}\pi t)\right)^2\right]$ as a function of observation length (time) of the mean calculation.

E. RMS Value of a Triangular Waveform

The *rms* value of a triangular waveform is known from calculus to be $V_m / \sqrt{3} = 0.5774 V_m$, and is of benefit in finding the level of ripple in a power supply. Here, we show this same result without using calculus.

A graph of a triangular waveform is shown below in Fig. 11(a) and its square is given in Fig. 11(b). Note that the mean of this squared waveform is the same whether the mean is computed over the full cycle or over just half the cycle. Hence, we will compute the mean from $-\frac{T}{4}$ to $\frac{T}{4}$. Note also that the equation of the triangular waveform over this time is given by

$$v(t) = \frac{4}{T} V_m t \qquad \text{for} \quad -\frac{T}{4} \le \text{t} \le \frac{T}{4}. \tag{18}$$

Therefore, the squared waveform has the equation

$$v^2(t) = \frac{16}{T^2} V_m^2 t^2 \quad \text{for} \quad -\frac{T}{4} \le \text{t} \le \frac{T}{4}. \tag{19}$$

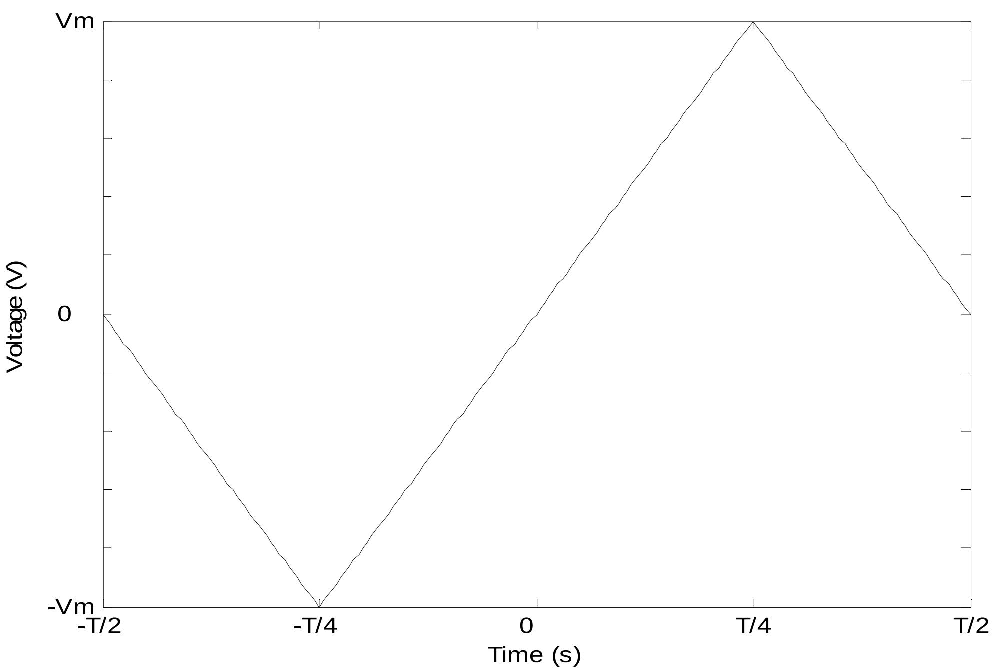

Fig. 11(a). Plot of a triangular waveform over a complete cycle.

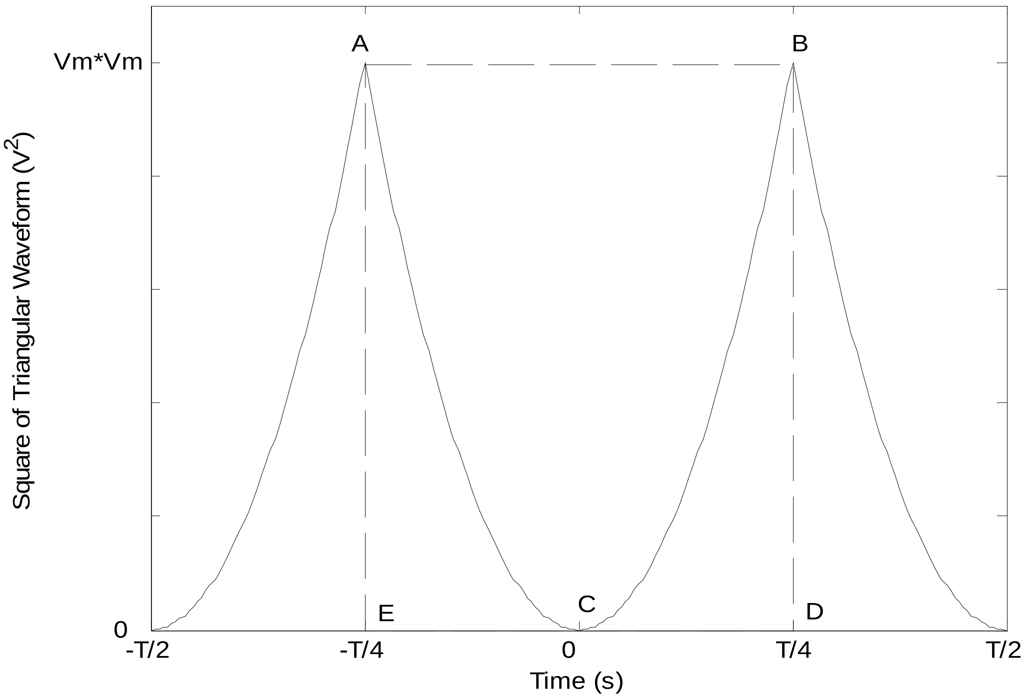

Fig. 11(b). Plot of the square of the triangular waveform above, over a complete cycle.

Furthermore, the squared waveform over this time forms the parabola ACB, whose height is V_m^2 and whose base is $T/2$. Therefore, the *enclosed* area of this parabola is well-known to be two-thirds the base times the height or $\frac{2}{3}\frac{T}{2}V_m^2$. However, the area *under* the squared waveform is the enclosed area of the parabola ACB subtracted from the enclosing rectangle ABDE, whose base is $T/2$ and whose height is V_m^2, and hence whose area is $\frac{T}{2}V_m^2$. So, the area below the curve of the squared triangular waveform is $\frac{1}{3}\frac{T}{2}V_m^2$, and therefore its mean is this value divided by $T/2$ or $\frac{1}{3}V_m^2$. Taking the square root of this gives the desired result.

F. RMS Value of a Pulse Train

We have left perhaps the easiest waveform for last. This waveform is a pulse train and is shown in Fig. 12(a), whereas its square is shown in Fig. 12(b). Clearly, the mean of the square is just the area of the squared voltage ($V_m^2\tau$) divided by the period, or $V_m^2\tau/T$. Hence, the *rms* value is given by $V_m\sqrt{\frac{\tau}{T}}$, which is a well-known result from calculus.

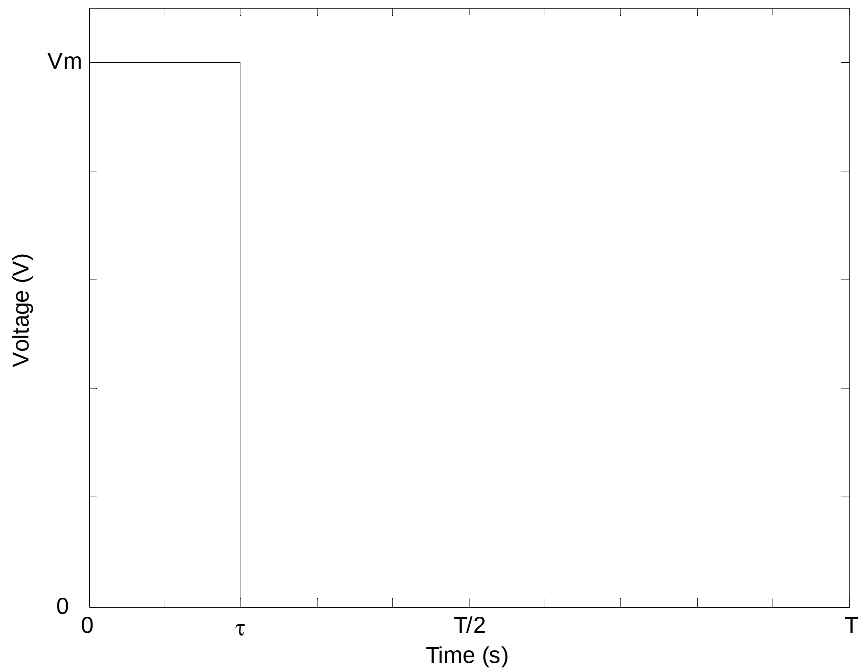

Fig. 12(a). Plot of a pulse train over a complete cycle.

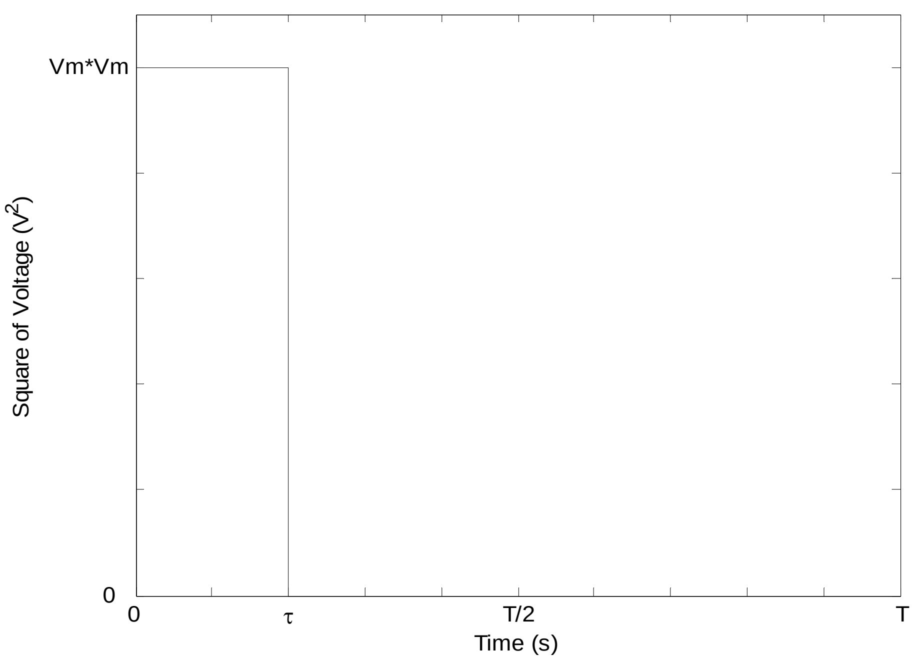

Fig. 12(b). Plot of the square of the pulse train above, over a complete cycle.

III. Conclusion

In this paper, the *rms* value of various waveforms was determined without using calculus. This should be of immense help to the Technology student. Furthermore, a discussion of the importance of using the correct observation length to compute the mean was also given.

References

[1] Guillermo Rico, "Tech Tip: Effective or RMS Voltage of a Sinusoid," the Technology Interface, Spring 2006, Vol. 6 No. 1 ISSN 1523-9926 http://engr.nmsu.edu/~etti/Spring06/ .

[2] Boylestad, R. L., *Introductory Circuit Analysis*, 11th edition, Prentice Hall, Upper Saddle River, NJ, 2007.