Derivation of the Exact Transconductance of a FET without Calculus

by

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Abstract - It is shown that the exact transconductance g_m of a field effect transistor (JFET, MESFET or MOSFET) can be derived without calculus. The method simply requires the solution of two simultaneous equations, one involving a quadratic equation and the other a linear equation.

I. Introduction

The transconductance g_m of a field effect transistor (FET) is used in the small signal analysis of FET circuits. It is well known that the transconductance at a particular bias point depends upon the slope of the tangent line of the transfer characteristic curve of the FET, at that point. Since slopes of tangent lines are conventionally found with calculus, most technology textbooks simply assume that calculus is necessary to derive the slope: hence, the transconductance is simply presented without derivation. However, it is known in the mathematical literature that slopes of tangent lines of some simple functions can be found without calculus. (See, for example, [1] and [2]: note the content of [1] is reproduced online in [3]). Unfortunately, this fact is not well known to engineering technologists. Hence, the purpose of this paper is to apply this mathematical result to the derivation of the slope of the transfer characteristic curve, i.e. the transconductance of a FET. This will be done for the JFET, the depletion and enhancement mode MESFET and MOSFET.

II. Transconductance of the JFET or Depletion Mode MESFET or MOSFET without Calculus

The transfer equation (Shockley's equation [4]) for the JFET or depletion mode MESFET or MOSFET is given by

$$I_D = I_{DSS}\left(1 - \frac{V_{GS}}{V_P}\right)^2, \tag{1}$$

where I_D is the drain current, V_P is the pinch-off voltage and V_{GS} is the gate-to-source voltage. Eq. (1) is plotted in Fig. 1.

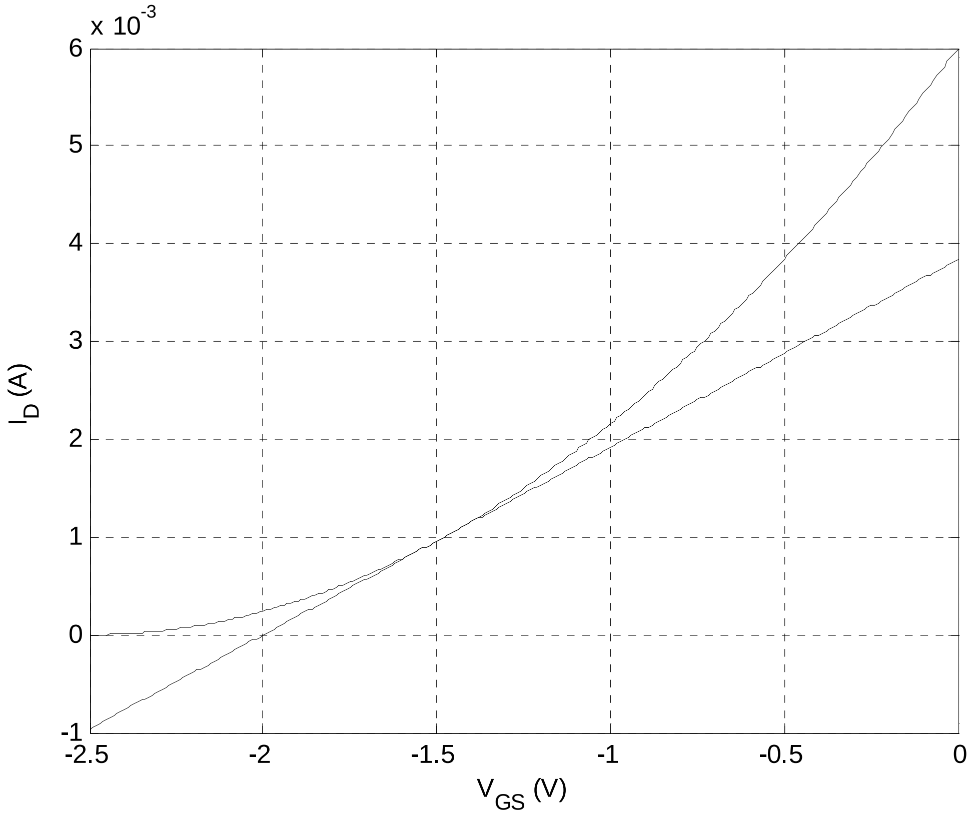

Fig. 1. Plot of the transfer equation of the JFET or depletion mode MESFET or MOSFET with $I_{DSS} = 6mA$ **and** $V_P = -2.5V$**. Also shown is the tangent line that touches the transfer curve at the point** $(V_{GS} = -1.5V, I_D = 0.96mA)$**. The slope of this tangent line is by definition the transconductance of the JFET or depletion mode MESFET or MOSFET, at this point.**

Note that the equation of the straight line also shown in Fig. 1 is given by

$$I_D = mV_{GS} + B, \tag{2}$$

where m is the slope of the line in A/V, i.e. Siemens, and B is the intercept of that line.

Furthermore, the straight line in (2) is a tangent line to the curve in Fig. 1 if it intercepts the curve at exactly one point. This particular definition of a tangent line is not a definition that works for any function. Fortunately, however, this definition is valid for the quadratic curve of Fig. 1, and other functions such as reciprocals, square roots and ellipses [1].

Hence, in order for (2) to be a tangent line to the quadratic curve, we must have

$$I_{DSS}\left(1-\frac{V_{GS}}{V_P}\right)^2 = mV_{GS} + B. \tag{3}$$

Equation (3) can be rewritten as

$$\begin{aligned}
&I_{DSS}\left(1-\frac{V_{GS}}{V_P}\right)^2 - mV_{GS} - B = 0 \\
&I_{DSS}\left(1-2\frac{V_{GS}}{V_P}+\frac{V_{GS}^2}{V_P^2}\right) - mV_{GS} - B = 0 \\
&\frac{I_{DSS}}{V_P^2}V_{GS}^2 - \left(\frac{2I_{DSS}}{V_P}+m\right)V_{GS} + I_{DSS} - B = 0 \\
&aV_{GS}^2 + bV_{GS} + c = 0,
\end{aligned} \tag{4}$$

where $a = \frac{I_{DSS}}{V_P^2}, b = -\left(\frac{2I_{DSS}}{V_P}+m\right)$ and $c = I_{DSS} - B.$

The quadratic formula can easily be applied to (4) to get

$$V_{GS} = -\frac{b}{2a} + \frac{\sqrt{b^2-4ac}}{2a} \text{ or } -\frac{b}{2a} - \frac{\sqrt{b^2-4ac}}{2a}. \tag{5}$$

Eq. (5) is stating that the straight line intersects the curve at two values of V_{GS}. However, in order for (2) to be a tangent, we require that there be only one point of intersection. Hence, we require that $\frac{\sqrt{b^2-4ac}}{2a} = 0$, or $b^2 = 4ac$, and $V_{GS} = -\frac{b}{2a}$. This means that

$$V_{GS} = -\frac{-\left(\frac{2I_{DSS}}{V_P} + m\right)V_P^2}{2I_{DSS}}, \text{or}$$

$$\frac{2I_{DSS}V_{GS}}{V_P^2} = \frac{2I_{DSS}}{V_P} + m, \text{or}$$

$$m = \frac{2I_{DSS}V_{GS}}{V_P^2} - \frac{2I_{DSS}}{V_P}, \text{or} \tag{6}$$

$$m = \frac{2I_{DSS}}{V_P}\left(\frac{V_{GS}}{V_P} - 1\right)$$

$$m = \frac{2I_{DSS}}{-V_P}\left(1 - \frac{V_{GS}}{V_P}\right).$$

As V_P is negative for n-channel FET, $-V_P = |V_P|$ and so (6) is quite often written as

$$m = \frac{2I_{DSS}}{|V_P|}\left(1 - \frac{V_{GS}}{V_P}\right). \tag{7}$$

Note that (7) is also valid for the p-channel FET.

Also, note that (7) is giving the slope of the straight line which is touching the curve of Fig.1 at exactly one point: that is, (7) is giving the slope of the tangent line of (1). This is in fact the definition of the transconductance of the JFET or depletion mode MESFET or MOSFET, i.e., $g_m = m.$ Hence, the transconductance of the JFET depletion mode MESFET or MOSFET has been derived without using calculus, which is thankfully the same expression that is derived with calculus.

III. Transconductance of an Enhancement Mode MESFET or MOSFET without Calculus

The transfer equation for the enhancement mode MESFET or MOSFET is given by

$$I_D = k\left(V_{GS} - V_T\right)^2, \tag{8}$$

where I_D is the drain current, V_{GS} is the gate-to-source voltage, V_T is the minimum gate-to-source voltage that is needed to turn the device on, $k = \frac{I_{DON}}{\left(V_{GSON} - V_T\right)^2}$, and $\left(V_{GSON}, I_{DON}\right)$ is a point on the curve when the device is on. Eq. (8) is plotted in Fig. 2 for representative values of k and V_T.

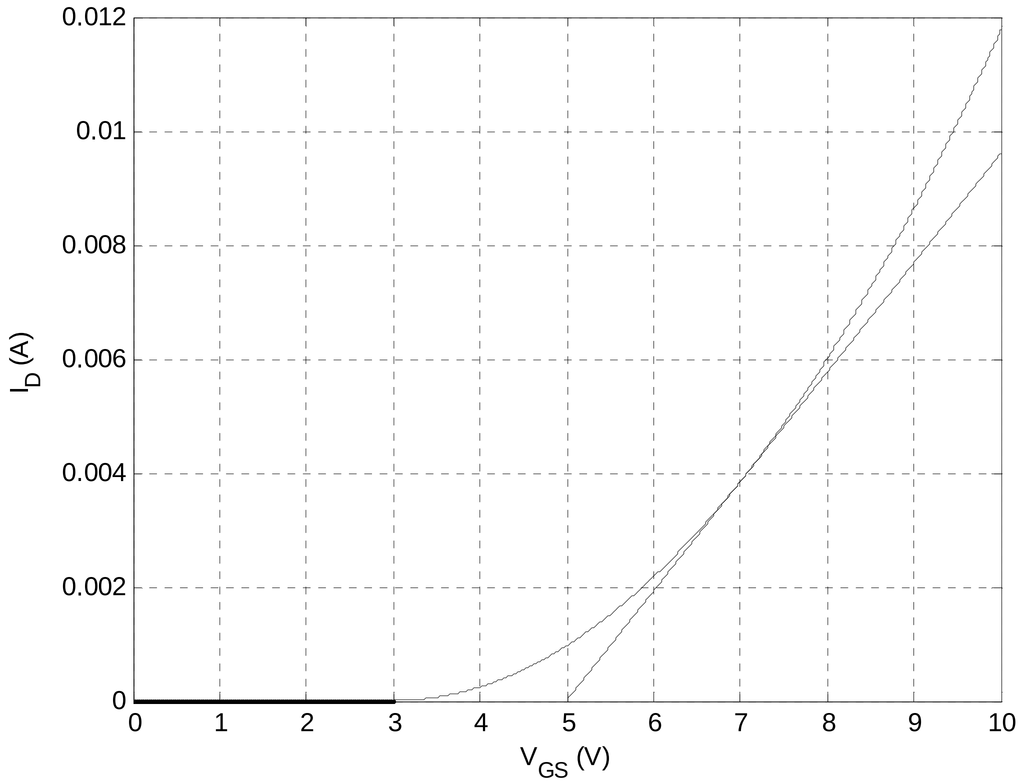

Fig. 2. Plot of the transfer equation of the enhancement mode MESFET or MOSFET with $k = 0.00024 A/V^2$ **and** $V_T = 3V$. **Also shown is the tangent line that touches the transfer curve at the point** $\left(V_{GS} = 7V, I_D = 3.84mA\right)$. **The slope of this tangent line is by definition the transconductance of the enhancement mode MESFET or MOSFET at this point.**

Note that the equation of the straight line also shown in Fig. 2 is given by

$$I_D = m_1 V_{GS} + B_1, \tag{9}$$

where m_1 is the slope of the line in A/V, i.e. Siemens, and B_1 is the intercept of that line.

Furthermore, the straight line in (9) is a tangent line to the curve in Fig. 2 if it intercepts the curve at exactly one point, as mentioned earlier. Hence, in order for (9) to be a tangent line to the quadratic curve, we must have

$$k\left(V_{GS} - V_T\right)^2 = m_1 V_{GS} + B_1. \tag{10}$$

Eq. (10) can be rewritten as

$$\begin{aligned} &k\left(V_{GS}-V_T\right)^2-m_1V_{GS}-B_1=0\\ &k\left(V_{GS}^2-2V_TV_{GS}+V_T^2\right)-m_1V_{GS}-B_1=0\\ &kV_{GS}^2-\left(2kV_T+m_1\right)V_{GS}+kV_T^2-B_1=0\\ &a_1V_{GS}^2+b_1V_{GS}+c_1=0, \end{aligned} \tag{11}$$

where $a_1=k, b_1=-\left(2kV_T+m_1\right)$ and $c_1=kV_T^2-B_1$.

The quadratic formula can easily be applied to (11) to get

$$V_{GS}=-\frac{b_1}{2a_1}+\frac{\sqrt{b_1^2-4a_1c_1}}{2a_1} \text{ or } -\frac{b_1}{2a_1}-\frac{\sqrt{b_1^2-4a_1c_1}}{2a_1}. \tag{12}$$

Eq. (12) is stating that the straight line intersects the curve at two values of V_{GS}. However, in order for (9) to be a tangent, we require that there be only one point of intersection. Hence, we require that $\frac{\sqrt{b_1^2-4a_1c_1}}{2a_1}=0$, or $b_1^2=4a_1c_1$, and $V_{GS}=-\frac{b_1}{2a_1}$. This means that

$$\begin{aligned} &V_{GS}=-\frac{-\left(2kV_T+m_1\right)}{2k}, \text{or}\\ &2kV_{GS}=2kV_T+m_1, \text{or}\\ &m_1=2kV_{GS}-2kV_T, \text{or}\\ &m_1=2k\left(V_{GS}-V_T\right). \end{aligned} \tag{13}$$

Notice that (13) is giving the slope of the straight line which is touching the curve of Fig.2 at exactly one point: that is, (13) is giving the slope of the tangent line of (8). This is in fact the definition of the transconductance of the enhancement mode MESFET or MOSFET, i.e., $g_m=m_1$. Hence, the transconductance of the enhancement mode MESFET or MOSFET has been derived without using calculus, which is thankfully the same expression that is derived with calculus.

VI. Conclusion

It has been shown that the transconductance of a JFET, MESFET or MOSFET can be derived without using calculus. This was possible because the transconductance is simply the slope of the tangent of the transfer characteristic curve, which happens to be a quadratic function. Happily, for such a function, the slope of the tangent line can be found quite easily by purely algebraic means, as has been demonstrated.

References

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Biography

KENNETH V. CARTWRIGHT (kvcartwright@yahoo.com) was born in Nassau, N. P., Bahamas. He received the B.E.Sc, M.S., and PhD degrees in electrical engineering from the University of Western Ontario in 1978, and Tulane University in 1987 and 1990, respectively. Presently, he is a professor of electrical engineering in the School of Mathematics, Physics and Technology and the Coordinator for the Electrical Engineering Technology Program, at The College of The Bahamas. His research interests include digital signal processing, analog and digital communication systems, and analog electronics. Some of this research is published in *IEEE Signal Processing Letters*, *IEEE Communication Letters*, *IEEE Transactions on Communications*, *IEEE* conferences' proceedings, and *the Technology Interface Journal*. Dr. Cartwright has also served as a reviewer for many journals. He is a member of Eta Kappa Nu, Tau Beta Pi, and numerous IEEE societies. (A more complete biography is available at **http://kvcartwright.googlepages.com/).**